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ABSTRACT

The purpose of this revised syllabus/guide is to help teachers and administrators plan and implement Course I in the State of New York Three-Year Sequence for High School Mathematics for Regents credit. Course I builds upon the State mathematics curriculum outlined in the Department's mathematics K-6, Grade 7, and Grade 8 publications. This guide includes the role of problem solving, the Course I outline, resources for teachers, a statement of Regents goals, and a discussion of the special needs of handicapped students. The course outline includes the following topics: (1) "Logic"; (2) "Algebra"; (3) "Geometry"; (4) "Analytic Geometry"; (5) "Probabilities"; and (6) "Statistics". Each section contains examples and sample problems related to that topic. (KR)

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Three-Year Sequence for High School Mathematics

Course I

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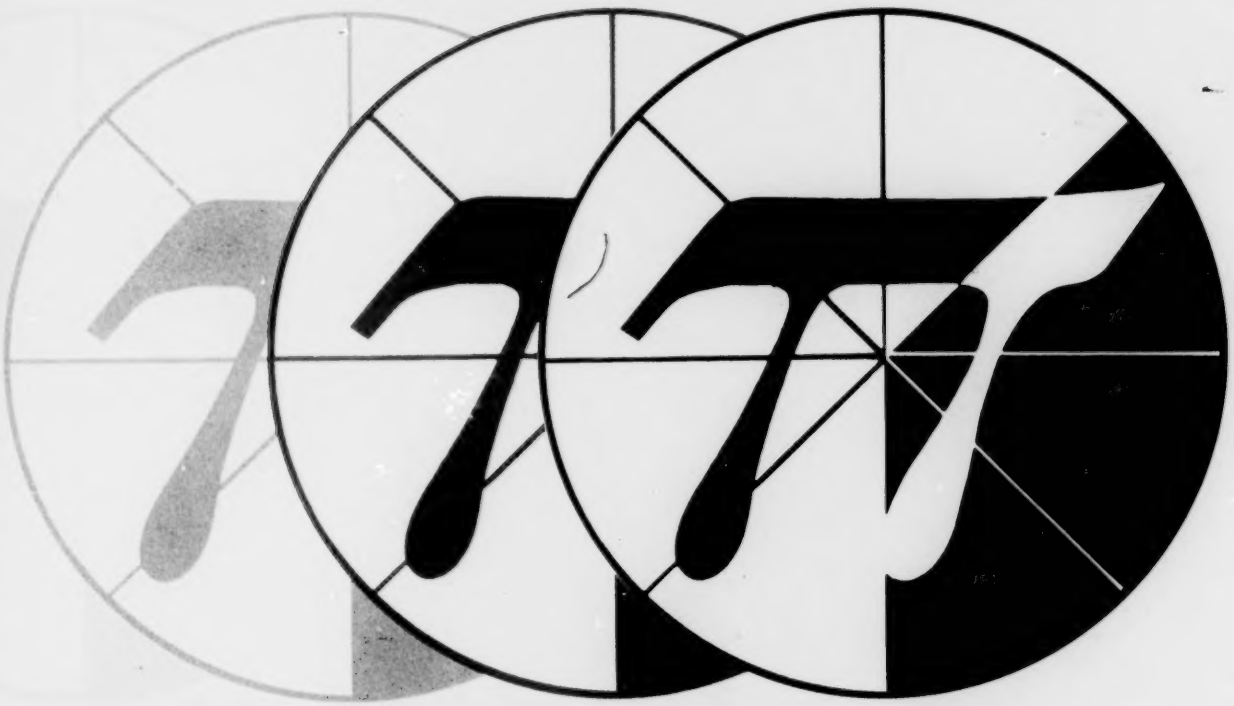

The University of the State of New York
The State Education Department
Bureau of Curriculum Development
Albany, New York 12234

Three-Year Sequence for High School Mathematics

Course I

The University of the State of New York
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Foreword

This syllabus is a revision of the 1976 document for the Three Year Sequence for High School Mathematics, Course I. Both the present syllabus and its predecessor clearly reflect the 1985 recommendation of the College Entrance Examination Board that the high school mathematics curriculum be organized so that the topics of computing, statistics, algebra, geometry, and functions are integrated into a three-year series of courses. The Board further states that an integrated curriculum can more easily incorporate applications of mathematics and mathematics modeling than can a curriculum divided by content.

In "An Agenda for Action, Recommendations for School Mathematics of the 1980s," the National Council of Teachers of Mathematics recommends that problem solving be the focus of school mathematics in the 1980s, where problem solving encompasses a multitude of routine and commonplace as well as nonroutine functions considered to be essential to the day-to-day living of every citizen. This revised syllabus is an attempt by the New York State Education Department to implement this recommendation.

In May 1984, an ad hoc committee was appointed by the Bureau of Mathematics Education to advise the Department concerning direction of the mathematics high school curriculum in preparation for a report to the Board of Regents. This committee met in June 1984, and consisted of the following:

Steven Adrian, Mahopac High School, Mahopac

Janet Burt, Clinton High School, Clinton

Walter Callahan, Shenendehowa Middle School, Elnora and President, Association of Mathematics Teachers of New York State, 1983-84

Bernard Fleishman, Rensselaer Polytechnic Institute, Troy

Joel Friedberg, Lehman High School, New York City and President, Mathematics Chairmen's Association of New York City, 1984-86

Jack Graver, Syracuse University, Syracuse

Meyer Jordan, Brooklyn College, Brooklyn

Irwin Kaufman, Director of Mathematics, New York City Board of Education

Edward Keenan, The Wheatley School, Old Westbury

Edith Luchins, Rensselaer Polytechnic Institute, Troy

Tim McMullin, Vestal Senior High School, Vestal

Lester Rubenfeld, Rensselaer Polytechnic Institute, Troy

Sister Ann Xavier, Nazareth Academy, Rochester

In June 1984, the State Education Department announced that the integrated sequence would become the official statewide program for high school mathematics, and that the program would be revised. A Revision Advisory Committee was appointed in July 1984 to begin the revision process. This committee has met regularly to review draft materials, and to provide input to the Department. It has included the following:

Janet Burt, Clinton High School, Clinton

David Call, Fort Ann Central School, Fort Ann

Roberta Eisenberg, Beach Channel High School, New York City

Pierre Fabinski, Christopher Columbus High School, New York City

Grahme Fischer, Grumman Aerospace Corporation, Bethpage

Anne Flansburg, Lafayette High School, Buffalo

Bernard Fleishman, Rensselaer Polytechnic Institute, Troy

Robert Frankel, Tottenville High School, New York City and President, Mathematics Chairmen's Association of New York City, 1986-88

Joel Friedberg, Herbert Lehman High School, New York City and President, Mathematics Chairmen's Association of New York City, 1984-86

Jack Graver, Syracuse University, Syracuse

Richard Klutch, Hunter College High School, New York City

Sharon Lanza, Grover Cleveland High School, Buffalo

Edith Luchins, Rensselaer Polytechnic Institute, Troy

Shelley Mora, Clara Barton High School, New York City

Fred Pedersen, Herricks High School, New Hyde Park

Lester Rubenfeld, Rensselaer Polytechnic Institute, Troy

George Stevens, Lansing High School, Lansing

Donald Thorpe, Vestal High School, Vestal

Sister Ann Xavier, Nazareth Academy, Rochester

Using the recommendations of the Advisory Committee, a Course I draft syllabus was organized and written by Sister Ann Xavier of Nazareth Academy, Richard Klutch of Hunter College High School, and Janet Burt of Clinton High School, working under the direction of Ben Lindeman and Lee Negus, Associates in the Bureau of Mathematics Education. Eleven school districts field tested the revised syllabus during the 1985-86 school year. These schools were:

Buffalo City School District

Dobbs Ferry Senior High School

The Wheatley School, East Williston

Fort Ann Central School

Herricks High School

Mahopac Senior High School

New York City School District
- Christopher Columbus High School
- Clara Barton High School
- Herbert Lehman High School
- Tottenville High School

Nottingham High School, Syracuse

Rochester City School District

South Colonie Central High School

Vestal Senior High School

Major changes in the Course I syllabus from the first printing include: (1) more specificity of topics in the syllabus, (2) an attempt to infuse problem solving throughout the program, and (3) an increase in basic algebraic skills. Minor changes in the syllabus include: (1) the transfer of constructions to Course II, (2) an inclusion of an informal intuitive introduction to geometric transformations, (3) inclusion of direct variation in conjunction with the equation of a straight line, (4) the finding of square roots, and (5) the transfer of the study of the central and inscribed angles in a circle to Course III.

Using recommendations from the Advisory Committee members and evaluations from the field test school teachers, the final draft of the syllabus was prepared and edited by Ben Lindeman and Lee Negus. Edward Lalor, Director of the Office of General Education Curriculum Development, represented the Bureau of Curriculum Development on the committee and served as the liaison with the Bureau of Mathematics Education. The final manuscript was prepared in the Bureau of General Education Curriculum Development.

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Goals and Objectives

The Statement of Regents Goals for Elementary and Secondary Students includes the following:

- Each student will master computation skills as a foundation to:
  - think logically and creatively
  - apply reasoning skills to issues and problems
  - perform basic mathematical calculations
  - determine what information is needed for particular purposes and be able to acquire, organize, and use that information for those purposes.
- Each student will use mathematics to gain knowledge, to learn methods of inquiry, and to use these in interdisciplinary applications.

These Regents goals act as a cornerstone for the K-6 and 7-8 mathematics syllabi. The Course I syllabus builds upon and extends this foundation.

The College Board, in "Academic Preparation for College," states that all college entrants will need to achieve proficiency in the following mathematical skills.

- the ability to apply mathematical techniques in the solution of real-life problems and to recognize when to apply those techniques
- familiarity with the language, notation, and deductive nature of mathematics and the ability to express quantitative ideas with precision
- the ability to use computers and calculators
- familiarity with the basic concepts of statistics and statistical reasoning
- knowledge in considerable depth and detail of algebra, geometry, and functions

It is the goal of the syllabi for Courses I, II, and III to provide students with a mathematical background which will be sufficient to meet their needs now and as they move into the twenty-first century.

The cultural aspects of mathematics should be made evident to learners so they can look upon it as a human invention important in itself-perhaps the greatest intellectual invention of the human mind. It has logic, abstraction, paradoxes, and patterns. These attributes should not be overlooked in mathematics instruction.

Mathematics can also be remarkably useful for dealing with real problems. Such problems should be an integral part of a student's high school mathematics program. Additionally, effective mathematics programs should help students develop an appreciation of mathematical ideas and thoughts, and should cultivate a trust and belief in results based upon mathematics. They should provide for:

- mathematical competence necessary to function and live independently in society
- mathematical competence in areas essential to function in the work an individual does or aspires to do
- mathematical competence needed to engage in hobbies, games, and leisure time pursuits that add enjoyment to life.

Mathematics should benefit students in three major goal areas:

- **affective goals**, including an appreciation of mathematical structure as a liberal art and an understanding of the role that mathematics has played as a part of our cultural heritage
- **content goals**, including knowledge, skills, and the language of mathematics
- **process goals**, including interpretation, application, deduction, and transfer of mathematical concepts and processes.

General cognitive goals for Course I include:

- the use of fundamental ideas of logic in developing logical reasoning and thinking skills as a means to reaching conclusions and as a prelude to more formal mathematical proof
- the mastery of algebraic skills for their unifying value in mathematical thinking and for their application in

practical formulas that appear in other disciplines, especially in the natural and social sciences

- an introduction to geometric ideas, including transformations, as a foundation for the more rigorous geometry to be encountered in later courses
- the development of a greater understanding of linear relations by studying their graphic representations in the coordinate plane
- the application of the techniques of probability theories studied in earlier grades through the analysis and solution of problems involving chance
- the utilization of the skills of collecting, organizing, and interpreting data to deal with the vast amount of information present in today's society
- the development of problem-solving strategies to solve a wide range of problems involving mathematical applications.

Specific content objectives for Course I are listed in the Course I outline of this syllabus and are described more fully in the teachers' notes that follow the Course outline. Instruction should be designed to integrate the topics with each other as much as possible, emphasizing the relations between the various segments of the curriculum. Problem solving should be infused throughout the course of study, with concepts and techniques from the various strands introduced as they contribute to an understanding and solution of a problem.

Introduction

This booklet presents an outline and some teaching notes for Course I of the Three-year Sequential Mathematics program of New York State. A major goal of this program is to spiral and sequence the topics included in the program throughout the three years so that the students review and build on previous learnings. A second objective is to integrate, when practical, the branches of mathematics, including the traditional topics of algebra, geometry, and trigonometry, and the additional topics of logic, probability, statistics, and transformational geometry. Suggestions for such integration are included throughout this publication. Although the descriptions of topics to be taught are presented here in a traditional unit format for the convenience of teachers and supervisors, teachers should integrate these topics in their own classrooms when possible. This integration, however, should be natural, not forced, thus benefitting both the instructional process and the learner outcomes.

It is estimated that the material presented here can be completed by most classes in one school year, usually the ninth grade. Some students, however, will complete the course in less than one year, while others will require more than one year. Likewise, although the course is written with the ninth grade students in mind, it is possible that some students may begin the material either in an earlier or a later year. Students with handicapping conditions who participate in this course may need instructional techniques or materials modified in order to meet the required objectives. Teachers providing such instruction should be aware of the needs of these students and modify approaches accordingly. Additionally, alternative testing modifications may need to be used with some students with handicapping conditions. For further information, see Appendix C, *Students with Handicapping Conditions*. It is expected that individual teachers will consider their own students in making judgments concerning time allotments, aspects of the material to be stressed, the degree of rigor employed with any topic, and extra material to be presented. It should be noted that teachers need not present the topics in the order given in the outline.

The inclusion of logic, probability, statistics, and transformational geometry, as well as the rearrangement of the topics of algebra and geometry constitutes a departure from tradition for many students and teachers. Perhaps, then, a brief statement on these areas is in order.

LOGIC: Logic is included here as a necessary, desirable, and enjoyable prelude to future work in mathematics, in other disciplines, and in everyday living. Implied in the Logic unit is the conviction that the logical process of reasoning is something that can and should be taught.

ALGEBRA: There is no preparation more vital to insuring a student's future success in mathematics than a firm, well-learned foundation of algebraic skills. The core of Course I is a solid introduction to the principles and techniques of algebra. Nonetheless, it will be noticed that some of the work with algebraic expressions traditionally found in a grade 9, Algebra I course has been omitted. Most of this work will appear, as needed, in succeeding years of the program. These topics have been postponed, not because this work in algebra is considered unimportant, but because its true importance is most clearly seen when it is placed in proximity with the problems it helps to clarify.

GEOMETRY: Some work with geometry, formerly a "tenth grade" topic, appears in Course I where it can illuminate, and be illuminated by, other mathematical concepts. Basically, the work with geometry in Course I is a relatively informal introduction to the more rigorous geometry encountered in Course II and Course III. It is believed that students can better appreciate the technique of formal proof if theorems to be proven involve geometric concepts with which they are already familiar. In addition to the traditional geometric topics, an intuitive introduction to transformations in the plane is presented. This informal introduction will pave the way for a more formal study of transformations in the coordinate plane in Courses II and III.

ANALYTIC GEOMETRY: The study of analytic geometry in Course I is primarily a study of the graphic representation of linear relations in the coordinate plane. The relationship between the algebraic representation of a linear relation and its graphic counterpart should be developed.

Students should be introduced to direct variation as an important application of the linear relation.

PROBABILITY AND STATISTICS: The degree to which our world is shaped and described by statistical inference is readily apparent. Therefore, we have an obligation to present to students the principles and terminology of this branch of mathematics. The probability and statistics in Course I are based on the material learned in elementary and junior high schools and lay the foundation for work in Course II and Course III.

In addition to the actual content of the Course I syllabus, the items listed below should be considered in the implementation of the syllabus.

ESTIMATION AND APPROXIMATION: Estimation and approximation are important strategies used in elementary and junior high schools and should be continued in Course I where appropriate. Estimation is often useful in areas such as finding square roots, or in probability activities, and should be systematically used when solving problems. Estimation, where appropriate, should be one of the steps in the problem-solving model (See "The Role of Problem Solving" pg. 6.). Students will not use estimation unless they find it useful; therefore, the teacher's role in teaching estimation should be one of modeling and motivating.

COMPUTERS AND CALCULATORS: There are no formal units on the computer or calculator in this syllabus. Nonetheless, the use of calculators, computers, and flow-charting is suggested and encouraged in those areas where such devices can contribute to mathematical understanding. These devices can also increase student participation and enjoyment and broaden the area from which applications of mathematical ideas can be selected. Individual schools are in the best position to judge how their own facilities, situations, and goals suggest the most reasonable place for electronic aids in their mathematics curriculum. Furthermore, individual teachers may want to give lesser or greater emphasis to the role which these modern components can play in the study of mathematics. It may be very helpful to use software and computer adaptations to assist students with handicapping conditions.

PROBLEM SOLVING: A major goal of any good mathematics curriculum must be to encourage students to think clearly and correctly, to analyze and deal with new situations, or, in short, to solve problems. Recently, the expression problem solving, variously defined, has come to stand for this broad intellectual skill. Problem solving in this broad sense, then, should be a part of every unit in this course and, ideally, in every teacher's lesson every day. In Appendix A of this booklet there is a review of some techniques used in problem solving. Teachers may wish to use this section for their own reference during the year, or may wish to review some of the techniques with their classes from time to time.

UNITS AND DIMENSIONAL ANALYSIS: When presenting problems that involve measurement, teachers should take care to include necessary units along with the numerical part of the measure. In science, engineering, and some industrial applications, it is necessary to change measurements from one unit to another. Dimensional analysis is a technique used to accomplish such conversions. This technique treats both a number and its unit of measure as factors. "Computations" are usually done with ratios where common units of measure are "divided out" in the same way common number factors are "divided out" or "canceled."

Although some consider this a process which is not mathematically correct, the symbolism has been developed by practitioners in such a way as to yield correct results. For example, if one wants to change 90 kilometers per hour to meters per second, the problem would be set up as follows:

$$\frac{90\text{ km}}{\text{hr}} \bullet \frac{1000\text{m}}{\text{km}} \bullet \frac{\text{hr}}{3600\text{ s}}$$

where $\frac{1000\text{m}}{\text{km}}$ and $\frac{\text{hr}}{3600\text{ s}}$ are each considered 1. Then by "dividing out" or "canceling" we get:

$$\frac{90\ \cancel{\text{km}}}{\cancel{\text{hr}}} \bullet \frac{1000\text{m}}{\cancel{\text{km}}} \bullet \frac{\cancel{\text{hr}}}{3600\text{ s}} = \frac{90000\text{ m}}{3600\text{ s}} = 25\,\frac{\text{m}}{\text{s}}$$

Examples such as this may help prepare students for their work in science.

INTERDISCIPLINARY ACTIVITIES: The Regents Action Plan recommends the use of special interdisciplinary projects to show students that the various subject areas are interrelated. Science, business, the social sciences (including economics), and technology education make extensive application of mathematical skills and concepts. Problem situations from these areas will show students the value of mathematics as a tool used in many human endeavors. A sample problem written by Bernard Fleishman, included in Appendix B, is an example of the type of classroom activity that is possible. It provides an example of the strong interaction between science and mathematics.

Music and art also may be used as vehicles to show the importance of mathematics in the arts. Much music theory is based upon basic principles of mathematics. Symmetry, tessellations, and form play a significant role in various art forms.

The role of pure mathematics cannot be overlooked when investigating the interdisciplinary aspects of mathematics. Major scientific theories of the 19th and 20th

centuries were able to be developed because the mathematics for them had been created before anyone saw their possible applications to science. For example, Einstein's general theory of relativity would not have been possible without the geometry of Riemann, invented in 1854, or without the theory of invariance developed by Cayley, Sylvester and other 19th century mathematicians. Many such examples could be given, and mathematics instruction should have as one of its objectives the goal to show, as Gauss said, that "Mathematics is Queen of the Sciences."

It is hoped that this program will give students a useful and enjoyable experience with mathematics which will encourage them to continue their mathematical studies. It is suggested that teachers give priority to a classroom atmosphere where there is ample time for student investigation, discovery, and discussion rather than emphasizing the task of "covering the material." Given sufficient time for the various possibilities these topics have to offer, students will participate in their first year of high school mathematics with a fuller appreciation of the fact that mathematics is a broader and more exciting field of study than many students often perceive it to be.

It is our belief that the sequential mathematics program begun here reflects what is happening in mathematics today. It is our hope that students who participate in it will be more likely to accept mathematics as an enjoyable and relevant part of their total experience and will want to continue with it, for their own benefit, and for the benefit of all of society.

The Role of Problem Solving

Problem solving should be an integral part of the learning activities for each unit. Problem-solving activities can be used in the classroom to stimulate discussion, to encourage creativity, and to foster a learning atmosphere in which students are encouraged to try a variety of techniques when confronting unfamiliar tasks, and to trust their abilities to think.

Problem solving serves several purposes in the classroom. A problem which uses the concepts and skills of a new unit may be presented to the students at the beginning of the unit to show them that they need additional skills to solve the problem. Hopefully, this will motivate them to learn the material in the unit. Business, industry, technology, and science provide realistic problems that generally interest students and show real-world applications of mathematics. Finally, problem solving provides a vehicle to use estimation and develop thinking skills as well as to review and reinforce previously presented concepts and skills.

Students should make a habit of estimating, where appropriate, the answer to a problem before solving it. If the solution is not "in the ballpark" of the estimated answer, the student should be aware to check his/her answer or estimate. The model given below includes estimation as one of the steps in solving a problem.

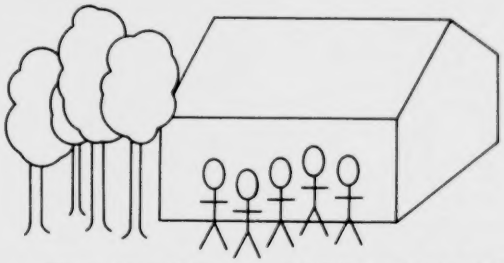
PROBLEM SOLVING MODEL:

1. Read and understand the problem.
2. Organize the information — diagrams, lists, etc.
3. Devise a plan for solving the problem.
4. Where appropriate, estimate the answer.
5. Carry out the plan.
6. Check the solution.

Teachers should familiarize themselves with the problem-solving activities of the 7th and 8th grade syllabi. A brief introduction to some of the principles used in solving problems at the junior high school level is contained in Appendix A of this document. As the material of this course is developed in the classroom, the strategies with which the students are already familiar should be utilized as a foundation for the algebraic methods that are an essential part of this course.

The study of logic presents a unique problem-solving situation. Students should be able to draw conclusions from information given in a problem in order to arrive at a solution.

EXAMPLE:



Three boys (Ben, Lee, and Rich) and two girls (Ann and Janet) pose for a yearbook picture in front of the school. From the clues below, can you tell where each student is standing?

1. No girl is at either end.
2. Ann is not standing next to Janet.
3. Rich is farthest from the trees.
4. Exactly two people are standing between Janet and Lee.

ANSWER: From left to right — Lee, Ann, Ben, Janet, Rich

After students learn the truth values of conjunctions, disjunctions, conditionals, and bi-conditionals, it will be possible for them to solve problems such as the one which follows. A problem such as this may be presented when the students begin their study of logic in order to demonstrate the need for unambiguous definitions of the connectives of logic or, after the study of logic, to demonstrate an application of what they have learned.

EXAMPLE:

Assume that each of the following statements is true.

If today is Wednesday, then Janet has gym. $W \rightarrow G$

Today is Monday or Wednesday.	M	V	W
If Janet does not have gym, then she will complete her essay for English.	∼ G	→	E
Today is not Monday.	∼ M		

From the truth value tables of compound sentences, it is possible to determine the truth value of the following statements:

Today is Wednesday.
Janet has gym.

It is not possible to determine the truth value of:
Janet will complete her essay for English.

∼ M is true (Given)
∴ M is false
M V W is true (Given)
∴ W is true
W → G is true (Given)
∴ G is true
and ∼ G is false
∼ G → E is true (Given)
∴ E may be either true or false

From the truth table for a conditional statement, if the antecedent is false, the conditional will be true whether the consequent is true or false.

Therefore, it is Wednesday and Janet has gym, but it is not possible to determine whether or not Janet completes her essay for English.

Solving problems algebraically is an important strategy to be learned in this course. The ability to use variables and equations (or inequalities) in translating a word problem into mathematical symbolism is an important competence for students. Problem solving can serve both as a tool for understanding mathematical ideas and as a strategy for finding the actual solution to a problem. Allowing students to first use a familiar non-algebraic strategy to solve a problem may help them to appreciate a more efficient, algebraic solution.

EXAMPLE:

In a school election there were three candidates. The winner received 5 more votes than the runner-up, and the runner-up received 3 more votes than the candidate who came in last. How many votes did each candidate receive if there were 92 votes cast?

Students may suggest a solution by trial and error for this problem.

If each candidate had received the same number of votes, each would have received 30 2/3 votes. Therefore, the winner received more than this.

Suppose that the winner received 33 votes.

Winner	Runner-up	Loser
33	28	25

This is a total of 86 votes. Since that total is 6 votes too few, increase each person's votes by 2.

Winner	Runner-up	Loser
35	30	27

This is a total of 92 votes. The winner has 5 more votes than the runner-up, and the runner-up has 3 more votes than the candidate who came in last. Therefore, this is the correct solution.

As the students learn to solve equations, a problem such as this can be solved by using an algebraic solution. The technique for any solution suggests the six steps given below:

1. Read and reread the problem.

A problem must be understood before it can be solved.

2. Organize the information.

Express the number of votes that each candidate received in terms of some variable, x.

Let x = the number of votes for the winner
$x - 5$ = the number of votes for the runner-up
$x - 8$ = the number of votes for the loser

3. Devise a plan.

Since we have decided to use an algebraic solution, our plan will involve the writing of an equation. Our equation uses the fact that the total number of votes cast was 92.

$x + (x - 5) + (x - 8) = 92$

4. Estimate.

Since the number of votes among the three candidates did not differ greatly, the number received by each candidate would be in the "neighborhood" of 1/3 of the total number of votes or approximately 30 votes.

5. Carry out the plan.

Solve the equation that we have written.

$$\begin{aligned} x + (x - 5) + (x - 8) &= 92 \\ 3x - 13 &= 92 \\ 3x &= 105 \\ x &= 35 \end{aligned}$$

6. Check the solution.

It is not enough to show that the solution makes the equation true. We must verify that the conditions of the original problem are, in fact, satisfied.

There is a total of 92 votes. The winner has 5 more votes than the runner-up, and the runner-up has 3 more votes than the candidate who came in last.

The winner received 35 votes.
The runner-up received 30 votes.
The loser received 27 votes.

Often the solution of a problem can be extended to gain new insights about the topic.

EXAMPLE:

Find two consecutive positive integers whose squares differ by 75.

$$\begin{aligned} \text{Let } x &= \text{the smaller integer} \\ x + 1 &= \text{the larger integer} \\ (x + 1)^2 - x^2 &= 75 \\ (x^2 + 2x + 1) - x^2 &= 75 \\ 2x + 1 &= 5 \\ 2x &= 74 \\ x &= 37 \\ 38^2 &= 1444 \text{ and } 37^2 = 1369 \\ 1444 - 1369 &= 75 \end{aligned}$$

Therefore, the consecutive integers are 37 and 38.

This problem can lead to new understandings of consecutive integers and their squares through questions such as the following:

1.a. Is the difference of the squares of two consecutive positive integers always equal to the sum of the integers?

$(x + 1)^2 - x^2$	=	$38^2 - 37^2$	=
$x^2 + 2x + 1 - x^2$	=	1444 − 1369	=
$2x + 1$	=	75	=
$x + (x + 1)$		37 + 38	

b. Find $93^2 - 92^2$ without squaring the integers.

2. Can the difference between the squares of two consecutive integers be even?

3. If it is known that the square of 46 is 2116, how can addition be used to find the square of 47?

4.a. What is the difference between the squares of two consecutive even integers or two consecutive odd integers?

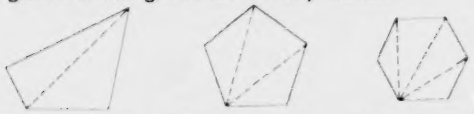
b. Can these differences be written in terms of the integers?

Geometric proof is a familiar type of problem solving in mathematics. In Course I, proofs should be intuitive rather than formal.

EXAMPLE:

What is the sum of the degree measures of the angles of a quadrilateral? Of a pentagon? Of a hexagon?

The use of a diagram will help the student suggest the use of the sum of the degree measures of the angles of a triangle to solve this problem.



$2(180°) = 360°$ $3(180°) = 540°$ $4(180°) = 720°$

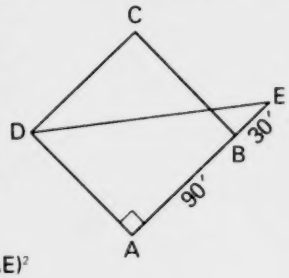
Students may be able to discover the general formula for the sum of the degree measures of the angles of a polygon of n sides, $(n - 2)180°$.

Many problems have geometric figures as their models and can be solved by algebraic strategies.

EXAMPLE:

A baseball diamond is a square with the distance from home plate to first base equal to 90 feet. The right fielder catches a ball on the first base line, 30 feet beyond first base, in the direction of right field. How far must she throw the ball to get it to third base?

A geometric diagram will assist the student in seeing that the Pythagorean theorem can be used to find the solution to the problem.



$$\begin{aligned} (ED)^2 &= (DA)^2 + (AE)^2 \\ &= (90)^2 + (120)^2 \\ &= 8100 + 14400 \\ &= 22500 \\ ED &= \sqrt{22500} = 150 \text{ feet} \end{aligned}$$

The right fielder would have to throw the ball 150 feet to third base.

Students should be encouraged to apply principles of geometry to concrete situations.

EXAMPLE:

Gillian had a rectangular picture which had a perimeter of 20 inches. After she had the picture enlarged, the area was 9 times that of the original. What is the perimeter of the enlarged picture?

Since the ratio of areas (1:9) is the square of the ratio of sides and the square of the ratio of the perimeters of similar figures, the ratio of perimeters is 1:3. If x represents the perimeter of the enlarged picture, then:

$$\frac{1}{3} = \frac{20}{x} \qquad x = 60$$

The perimeter of the enlarged picture is 60 inches. After studying irrational numbers, the problem may be changed so that the enlargement has an area 8 times that of the original. Then the ratio of sides and ratio of perimeters would be 1: $\sqrt{8}$ or 1:$2\sqrt{2}$.

The ability to confront nonroutine tasks, that is, to deal successfully with those tasks that require the selection of a procedure that is not immediately obvious, is a desirable outcome of education in mathematics. This skill does not de-

velop automatically with the mastery of computational skills and techniques, but must be developed through practice, through the sharing of insights, and through the development of the habits of questioning and curiosity.

As teachers of mathematics, we see mathematics as a logical, coherent set of concepts that develop naturally one from another. Many of our students do not see it that way because, in the interests of "covering the course" or "completing the work of a particular year," mathematics has often been presented as a set of rules to be learned rather than as ideas that unfold or emerge from the learner's ability to see and appreciate relationships. Time given to problem solving can be time well spent because the enthusiasm and excitement which can be generated leads to more retentive learning.

The purpose, then, of this introductory section is to encourage the use of nonroutine problems throughout the course in addition to those problems that are part of the content of each unit. Problems may serve the following purposes:

1. to motivate the introduction of a new topic or the development of a new skill or technique;
2. to provide practice with a skill that has been, or is being, developed;
3. to allow the exploration of concepts beyond those suggested by the content of the course;
4. to develop in the individual student the confidence to try various techniques when faced with an unfamiliar problem.

Teachers should strive to use activities which achieve each of these purposes.

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A Note on Time and Sequence

It is difficult to supply a meaningful time frame for an integrated course since it is intended that topics interact with each other in the development of the material. The following chart should be used only as a general frame of reference.

Topic	*Days*
I. Logic	10–15
II. Algebra	55–65
III. Geometry	35–45
IV. Analytic Geometry	15–20
V. Probability	5–10
VI. Statistics	5–10

Depending on the students' backgrounds, time allotments for one or more of these topics may vary significantly.

It should be noted that the time suggestion is not intended to imply that these topics should be taught in this order or as separate blocks of material. Algebraic techniques will be used throughout the course in the development of other topics.

I.

Logic

The study of the Logic unit should give the student a background in drawing sound conclusions from given information. Mathematics is a language, often expressed symbolically. The student's work in logic should help develop problem-solving skills in analyzing problems, translating written statements into symbolic statements, organizing statements in an orderly fashion, and developing correct conclusions based on the principles of logical reasoning. It is recommended that the Logic unit be early in the course and that examples from areas of mathematics (geometry, arithmetic, number theory) as well as examples from the student's everyday experiences be incorporated into the logic problems.

TRUTH VALUE OF SIMPLE SENTENCES

Students should be able to state whether simple declarative sentences are true or false. Examples should be chosen such that truth values can be determined objectively and do not depend upon the opinion of the reader. For example, we cannot establish the truth value of a sentence such as "Spinach is delicious.", since its truth value depends upon the subjective feeling of the reader.

Closed Sentences

A closed sentence is an objective statement which is either true or false.

EXAMPLES:

(1) Albany is the capital of New York State. (True)
(2) No prime number is even. (False)
(3) The sum of two negative numbers is a negative number. (True)

It is possible that a closed statement will have a different truth value at different times. For example, "Tomorrow is Tuesday." is true if today is Monday, but is false if today is any other day of the week.

Open Sentences

An open sentence is a statement which contains pronouns or variables and becomes either true or false upon the replacement of those pronouns or variables.

EXAMPLES:

(1) x is divisible by 3
(2) $a + 7 = 11$
(3) He was the 36th President of the U.S.
(4) $y - 3 < 10$
(5) $x + 1 = x$

Sentence 1, for example, is true if x is replaced by 12, but false if x is replaced by 11.

(a) The set from which replacements are made is called the replacement set, the universe, or the domain. For example, the replacement set of sentences 1, 2, 4,

and 5 could be the set of real numbers. The domain of sentence 3 could be the set of all U.S. Presidents or the set of all 20th century Americans.

(b) The members of a replacement set of an open sentence which make the sentence true constitute the solution set of the open sentence.

(c) A solution set may contain one, many, or no members. For example, the solution set for sentence 3 contains only John F. Kennedy, whereas the solution set for sentence 1 contains any multiple of 3: {3, 6, 9, 12, . . .}. This solution set contains an infinite number of members. The solution set of sentence 5 contains no value since every real number makes the sentence false. The solution set is empty and is indicated by $\emptyset$ or { }.

(d) If the replacement set for sentence 4 is the set of integers, then the solution set is {14, 15, 16, . . .}. If, on the other hand, the replacement set is the set of real numbers, the solution set contains all values of y such that $y > 13$.

Symbolism is notation introduced into any area of mathematics to aid understanding and communication of ideas. Statements in logic are designated by single letters, p, q, r, etc.

Negations

The negation of a statement p is "not p"; in symbols, $\sim p$. The truth value of $\sim p$ is the "opposite" of that of p. Thus, $\sim p$ is false when p is true and $\sim p$ is true when p is false. This can be summarized in the following table, called a truth table.

p	$\sim p$
T	F
F	T

Students should have practice forming negations in words. A negation can be formed by prefacing the statement with the words, "It is not so that . . . ." Since the resulting statement often sounds stilted and cumbersome, it will be easier to form most negations by changing the verb of the sentence.

EXAMPLES:

p: Albany is the capital of New York State.
$\sim p$: Albany is not the capital of New York State.
q: The product of two negative numbers is a negative number.
$\sim q$: The product of two negative numbers is not a negative number.
r: $7 < 5$
$\sim r$: $7 \not< 5$ (Note: This is not the same as $7 > 5$. An equivalent statement is $7 \geq 5$.)

It should be noted that the negation of the negation of p is p, i.e., $\sim(\sim p)$ is p.

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

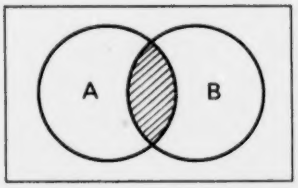
TRUTH VALUE OF COMPOUND SENTENCES

Conjunction

A conjunction is a compound statement formed by joining two statements with the connective AND. "p and q" is symbolized by $p \wedge q$.

The truth value of a conjunction of two statements is true if and only if BOTH statements are true. Otherwise, the conjunction is false. This definition may be summarized in the following truth table: (Note: There are four possible arrangements of the truth values of p and q.)

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F



The similarity between the definition of conjunction and the intersection of two sets, $(A \cap B)$, may be used to illustrate the truth values of a conjunction.

Students should be given many opportunities to write conjunctions and to determine the truth values of the conjunctions, using both closed and open sentences.

EXAMPLE 1:

p: The LCM (least common multiple) of 4 and 6 is 12.
q: The GCF (greatest common factor) of 12 and 16 is 2.
p ∧ q: The LCM of 4 and 6 is 12 and the GCF of 12 and 16 is 2.

Since p is true and q is false, p ∧ q is false.

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

EXAMPLE 2:

r: x is odd
s: x is prime
r ∧ s: x is odd and x is prime

The truth value of r ∧ s depends on the value of x.

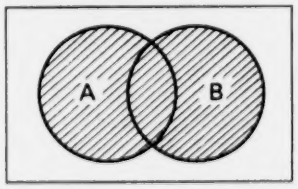
r	s	r ∧ s	**Examples:**	Truth Value of Conjunction
T	T	T	If x = 3, then both r and s are true	true
T	F	F	If x = 9, r is true, s is false	false
F	T	F	If x = 2, r is false, s is true	false
F	F	F	If x = 6, both r and s are false	false

Disjunction

A disjunction is a compound statement formed by joining two statements with the connective OR. "p or q" is symbolized by p ∨ q.

The truth value of the disjunction of two statements is true if EITHER OR BOTH statements are true and is false only if both statements are false. When the word "or" is used to mean that one or both of the statements are true, it is called the inclusive or. This definition, using the inclusive or, may be summarized in the following truth table:

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F



The exclusive or, meaning that one and only one of the statements is true, has a different truth table than the one shown and will not be considered in this course.

Again, the similarity between the definition of disjunction and the union of two sets ($A \cup B$) may be used to illustrate the truth values of a disjunction.

Students should be given many opportunities to write disjunctions and to determine the truth values of disjunctions, using both closed and open statements.

EXAMPLE 1:

p: the square of a positive number is positive
q: the square of a negative number is negative
p V q is a true statement since p is true.

EXAMPLE 2:

p: x is divisible by 2
q: x is divisible by 3

The truth value of p V q depends upon the value of x.

p	q	p V q	**Examples:**	Truth Value of Disjunction
T	T	T	If x is 6, both p and q are true	true
T	F	T	If x is 8, p is true, q is false	true
F	T	T	If x is 15, p is false, q is true	true
F	F	F	If x is 11, both p and q are false	false

EXAMPLE 3:

r: x is odd
s: x is prime
r V s: x is odd or x is prime

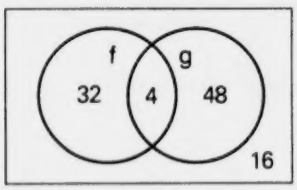
The truth value of r s depends on the value of x.

r	s	r V s	**Examples:**	Truth Value of Disjunction
T	T	T	If x = 3, both r and s are true.	true
T	F	T	If x = 9, r is true, s is false.	true
F	T	T	If x = 2, r is false, s is true.	true
F	F	F	If x = 6, both r and s are false.	false

The ideas of conjunction and disjunction can be applied to problems such as the following:

In a class of 100 students, 36 take French, 52 take German, and 4 take both languages. How many students are not enrolled in either language?

Solution — the use of a Venn diagram to show the number of students in each language and in both, facilitates the solution of the problem.



f: 36 students enrolled in French
g: 52 students enrolled in German
f Λ g: 4 students enrolled in both
f V g: 84 students enrolled in either French or German
∼(f V g): 16 students enrolled in neither

Conditional

The conditional, also referred to as an implication, is a compound statement in the form: "If p, then q" or "p implies q," symbolized by $(p \rightarrow q)$. The first statement, p, is called the hypothesis or antecedent, and the second statement, q, is called the conclusion or the consequent.

Some authors distinguish between the terms "conditional" and "implication" by defining an implication as a true conditional, but it is suggested that this distinction not be probed at this time. Students may, however, confuse the truth of a conditional with the truth of its conclusion, and it is suggested that some attention be given to this possible difficulty.

The conditional is defined to be true unless a true hypothesis leads to a false conclusion. This is summarized in the following truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Unlike the truth values of the conjunction and disjunction, the truth values of the conditional may not seem to follow the "common sense" of the student. The student may find the truth values in the case of a false hypothesis difficult to accept. The teacher can motivate the understanding of the definition by using several examples. One example, using the idea of a promise kept or broken, is:

Suppose Mr. Klutch makes the following promise to his class: "If you complete an extra credit computer program, then you will receive 5 extra points on your average." The chart which follows illustrates the 4 possible situations:

Student completes extra credit	Mr. Klutch adds 5 points on average	Mr. Klutch keeps promise (does not break promise)
YES	YES	YES
YES	NO	NO
NO	YES	?
NO	NO	?

In the latter two cases, the students may feel that they cannot assign a truth value since Mr. Klutch made no promise regarding what he would do if the program were not completed. However, it should be pointed out that a promise is kept until broken. In that sense, since the promise is not broken in either of the latter cases, the conditional can be considered true. In fact, the only way to break the promise is in line 2 where Mr. Klutch failed to add 5 points after the student completed the project.

An example of a true conditional with both a false premise and a false conclusion is:

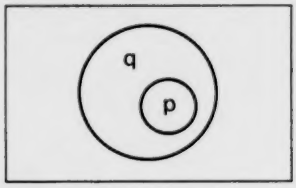
If there is no even prime, then 2 is not a prime.

The conditional statement is one of the most frequently used statements in deductive reasoning. Often the hypothesis of the conditional does not precede the conclusion when written in words. Students need to work with conditionals stated in various ways. They need many opportunities to write conditionals as sentences and to decide on the truth values of the compound statements.

The conditional, $p \rightarrow q$, may be stated in words as:

if p, then q
q if p
p only if q

The use of a Venn diagram may help students visualize the equivalence of the 3 forms of the conditional.



EXAMPLE 1:

p: A number is divisible by 6.
q: A number is divisible by 2.
The following are equivalent statements translating $p \rightarrow q$:
If a number is divisible by 6, then the number is divisible by 2.
A number is divisible by 2 if it is divisible by 6.
A number is divisible by 6 only if it is divisible by 2.

EXAMPLE 2:

r: ABCD is a rectangle.
s: ABCD is a parallelogram.
$r \rightarrow s$ can be translated as:
If ABCD is a rectangle, then ABCD is a parallelogram.
ABCD is a parallelogram if ABCD is a rectangle.
ABCD is a rectangle only if ABCD is a parallelogram.

Related Conditionals

For every conditional ($p \rightarrow q$), there are 3 related conditionals with which students should be familiar.

ORIGINAL	$p \rightarrow q$
CONVERSE	$q \rightarrow p$
INVERSE	$\sim p \rightarrow \sim q$
CONTRAPOSITIVE	$\sim q \rightarrow \sim p$

Students should investigate the relationships between the truth values of the related conditionals through both examples and truth tables. Statements should be chosen which clearly illustrate that the converse and inverse do not necessarily have the same truth value as the original. The contrapositive will, however, always have the same truth value.

EXAMPLES:

If a woman is a U.S. Senator, she is a member of Congress.
If a number is divisible by 10, the number is divisible by 5.
If a figure is a square, the figure is a parallelogram.
If an animal is a collie, then it is a dog.

There are many opportunities to show the role of the related conditionals in everyday life. Advertising slogans are often designed with the intent that the customer will assume the converse or the inverse is true. An example might be:

If you eat Super Toasties, you will have a nourishing breakfast.

The manufacturer expects (and hopes) some consumers will infer the inverse, i.e., "If you do not eat Super Toasties for breakfast, you will not have a nourishing breakfast."

Examples may also be found in political slogans. Students can find statements in newspapers, magazines, political flyers, etc., illustrating the use of related conditionals.

A typical political advertisement might say, "If you elect Jones as Mayor, your taxes will go down."

The hope is that the public will assume the truth of the inverse, i.e., "If you do not elect Jones as Mayor, your taxes will not go down."

In addition to examples from everyday life, many examples of related conditionals can be found from the properties of geometric figures. Such examples include:

If two sides of a triangle are congruent, the angles opposite these sides are congruent. (The original, the converse, the inverse, and the contrapositive are all true!)

If two angles are right angles, the angles have equal measure. (The converse and inverse are not true!)

Bi-conditional

The bi-conditional, or equivalence as it is sometimes called, is a compound statement in the form: "p if and only if q," symbolized by ($p \leftrightarrow q$).

Again, a distinction may be made between a bi-conditional and the equivalence by defining a true bi-conditional as an equivalence. It is suggested that this distinction not be probed at this time.

The bi-conditional is defined to be true whenever both parts have the same truth value. This is summarized in the following truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

It should be pointed out that the statement $p \leftrightarrow q$ is a combination of the statements $p \rightarrow q$ and $q \rightarrow p$. These statements can be written "p only if q, and p if q," or can be combined to "p if and only if q," often abbreviated as "p iff q."

In later units of this course, the teacher should call attention to the use of the bi-conditional in stating definitions and some theorems. Some examples are:

The isosceles triangle theorem in geometry can be stated, "Two angles of a triangle are congruent if and only if two sides of the triangle are congruent."

The Pythagorean theorem can be stated, "A triangle is a right triangle if and only if the square of the longest side is equal to the sum of the squares of the other two sides."

In addition to these examples, definitions in mathematics are often written using the bi-conditional. For example:

A number is prime if and only if the number has exactly two factors, one and itself.

"$a - c = b$ if and only if $a = b + c$" is the definition of subtraction.

TAUTOLOGIES

A compound statement which is always true regardless of the truth value of the individual statements is called a tautology.

Proof Using Truth Tables

Students should be able to construct truth tables for compound statements to prove whether or not the statements are tautologies.

EXAMPLE:

Show: $[(p \rightarrow q) \wedge p] \rightarrow q$ is a tautology.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The final column shows the compound statement is true for all combinations of the truth values of p and q. Hence, the compound statement is a tautology. This particular compound statement is an important Law of Inference which students will use in Course II.

EXAMPLE:

Show: $\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$ is a tautology.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

Again the final column shows the statement is a tautology and shows how to negate a conjunction.

Other laws of reasoning which provide good examples for tautology exercises include:

$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$

$[(p \vee q) \wedge \sim p] \rightarrow q$

$\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$

Using a truth table, the students should now be able to prove that the bi-conditional is the conjunction of a conditional and its converse.

$(p \leftrightarrow q) \leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$

Truth tables can be constructed to demonstrate that a conditional and its converse are not equivalent, i.e.,

$(p \rightarrow q) \leftrightarrow (q \rightarrow p)$ is NOT a tautology.

Applications

In addition to constructing truth tables to demonstrate tautologies, students should be able to draw conclusions from a set of given statements either by using patterns established by tautologies or by assigning truth values to the individual statements.

EXAMPLE 1:

If Jane is a freshman, then she takes Spanish I.
Jane is a freshman.

If these two statements are true, can we conclude that Jane takes Spanish I?

If J represents "If Jane is a freshman" and S represents "Jane takes Spanish I," these statements can be represented symbolically by:

$J \rightarrow S$

J

$\therefore S$

If the student recognizes the form of this pair of statements as a tautology previously proven $[(p \rightarrow q) \wedge p] \rightarrow q$, he may immediately conclude S, i.e., "Jane takes Spanish I."

Another method of solution follows:

$J \rightarrow S$ is true

$\underline{J\quad\quad}$ is true

Since J is true in the second statement, it is also true in the first statement. Since the first statement is true and its hypothesis is true, its conclusion must be true. The students may record the truth value next to the symbols, as follows:

$J^{(T)} \rightarrow S^{(T)}$

$\underline{J^{(T)}\quad\quad}$

$S^{(T)}$

EXAMPLE 2:

If ABCD is a square, then ABCD is a parallelogram. ABCD is not a square.

If these two statements are true, can we determine whether or not ABCD is a parallelogram?

If s represents "ABCD is a square" and p represents "ABCD is a parallelogram," these statements can be represented symbolically by:

$s \rightarrow p$

$\underline{\sim s\quad\quad}$

?

Since $\sim s$ is true, s is false. The first statement is true if the premise is false, regardless of the truth value of the conclusion. Therefore, the truth value of p cannot be ascertained.

There are many areas of mathematics (number theory, geometry, arithmetic concepts, etc.) which contain material with good examples for students to practice their skills in logical reasoning.

EXAMPLE 1:

If x is divisible by 9, then x is divisible by 3.
x is not divisible by 3.
$\therefore$ x is not divisible by 9.

EXAMPLE 2:

Either x is even or x is odd.
x is not even.
$\therefore$ x is odd.

Classroom discussions may include more complex problems such as the following:

Tom will attend either NYU or Penn State.
If Tom attends NYU, he will major in mathematics.
Tom does not attend Penn State.

If these statements are true, then the following are some conclusions which can be drawn

Tom will attend NYU.
Tom will major in mathematics.
Tom will attend NYU and will major in mathematics.

Examples such as these should be considered preparatory for Course II work; mastery of these and comparable applications is *not* expected for the Course I Regents.

II.

Algebra

The study of algebra provides students with the skills that will be useful as they continue to study mathematics. Algebra is a tool that allows the user to explore relationships in more general terms, and is a language that enables the user to express ideas in symbolic form.

This unit outlines specific objectives and the relationship of these objectives to the course as a whole.

OPERATIONS ON ALGEBRAIC EXPRESSIONS

Review of Operations with Signed Numbers

The student should be able to add, subtract, multiply, and divide signed numbers, including those expressed as decimals or common fractions. The use of the commutative, associative, and distributive properties is basic to the operations on the real numbers and should be employed throughout the discussion.

EXAMPLE:

$$\begin{aligned} -0.2(-3.4 + 7.2) &= -0.2(-3.4) - 0.2(7.2) \\ -0.2(3.8) &= \quad 0.68 \quad - 1.44 \\ -0.76 &= \quad -0.76 \end{aligned}$$

Operations with signed numbers can be reviewed in conjunction with other learning activities, such as the truth tables of logic.

EXAMPLE:

Give the truth value of: $-2 - 7 = +9$ and $-2(-7) = +14$	Since $-2 - 7 = +9$ is false, this conjunction is false.
If $-6/-3 = +2$, then $-3(+2) = -6$	Since the hypothesis and conclusion are both true, the conditional is true.

Use of Variables

Students should understand how expressions such as $2x$, $x + 3$, and $2x - 5$ are related to the variable x. When a number is represented by a variable, students should be able to use that variable to write other expressions such as "twice the number," "three more than the number, or" the "number divided by six" in terms of the given variable.

EXAMPLE 1:

The length of a rectangle is represented by x.
Describe, in words, the relationship of the width to the length if the width is represented by

$x - 7$: width is 7 less than the length
$x/2$: width is half the length
$2x - 12$: width is 12 less than twice the length

EXAMPLE 2:

Jennifer's age can be represented by x. Express, in terms of x, Jennifer's age:

in 10 years	$x + 10$
5 years ago	$x - 5$
when she is twice as old as she is now	$2x$

The students should be able to make up problems which use expressions such as x, 2x, and $2x - 1$.

The evaluation of algebraic expressions provides additional opportunities for practice with signed numbers. Students should know that algebraic expressions are evaluated in the following order:

1. operations enclosed in parentheses
2. powers and roots
3. multiplications and divisions
4. additions and subtractions

Special attention should be paid to parentheses which are used to change the order of operations. Operations enclosed in parentheses precede all other operations.

EXAMPLE 1:

Find the value of the given expressions when $a = -3$ and $b = 12$.

$-5a^2 + b/4$	$-5(a + b/4)$
$-5(-3)^2 + 12/4$	$-5(-3 + 12/4)$
$-5(+9) + 3$	$-5(-3 + 3)$
$-45 + 3$	$-5(0)$
-42	0

EXAMPLE 2:

Find the value of the given expression when $x = -2$ and $y = 3$.

xy^2	$(xy)^2$
$-2(3)^2$	$[-2(3)]^2$
$-2(9)$	$(-6)^2$
-18	36

Formulas from many areas, such as geometry, science, and business, provide suitable practice in evaluating algebraic expressions, while at the same time providing an opportunity for students to learn or review the formulas.

EXAMPLE:

The formula $C = 5/9(F - 32)$ converts degrees Fahrenheit (F) to degrees Celsius (C). Find the Celsius temperature when the Fahrenheit temperature is 12 degrees.

This problem provides an opportunity for the teacher to discuss estimation and approximation. Since 12°F is below the freezing point of water, the converted temperature should be below zero (or negative), as 0°C is the freezing point of water on the Celsius scale.

$$\begin{aligned} C &= 5/9(F - 32) \\ &= 5/9(12 - 32) \\ &= 5/9(-20) \\ &= -100/9 \\ &= -11\ 1/9 \end{aligned}$$

Addition of Polynomials

Fundamental operations with monomial expressions are the foundations of future work with equations and problem solving. Since the variables used in algebraic expressions repre-

sent real numbers, the commutative, associative, and distributive properties can be applied in simplifying these expressions.

The use of the distributive property for multiplication over addition can help students understand addition and avoid common errors.

EXAMPLE 1:

$3x + 7x = (3 + 7)x = 10x$
$3x + 7x \neq 10x^2$

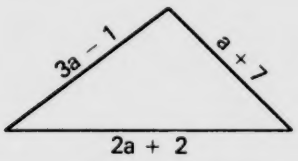
EXAMPLE 2:

$3x - x = 3x - 1x = (3 - 1)x = 2x$
$3x - x \neq 3$

A polynomial is a monomial or the indicated addition of two or more terms. The sum of two polynomials is in simplest form when no two terms are like terms.

Students should be able to express the perimeter of a polygon as the sum of the measures of the sides. The use of variables to represent the measures of the sides of a polygon can provide practice in adding polynomials.

EXAMPLE:



Express, in terms of a, the perimeter of a triangle if the measures of the sides are $2a + 2$, $3a - 1$, and $a + 7$.

$P = (2a + 2) + (3a - 1) + (a + 7)$
$P = 2a + 2 + 3a - 1 + a + 7$
$P = 6a + 8$

The use of the distributive property to add monomials is also basic to the addition of fractions.

EXAMPLES:

$$\frac{5}{x} + \frac{3}{x} - \frac{1}{x} = 5\left(\frac{1}{x}\right) + 3\left(\frac{1}{x}\right) - 1\left(\frac{1}{x}\right) = (5 + 3 - 1)\left(\frac{1}{x}\right) = \frac{7}{x}$$

$$\frac{2x}{7} + \frac{4}{7} = 2x\left(\frac{1}{7}\right) + 4\left(\frac{1}{7}\right) = (2x + 4)\left(\frac{1}{7}\right) = \frac{2x + 4}{7}$$

Until students have studied the multiplication of algebraic expressions, adding fractions should be limited to those with like monomial denominators.

Multiplication of Polynomials

Students should understand the meaning of a positive, integral exponent and be able to use exponents correctly in multiplying powers with like bases.

$x^a \cdot x^b = x^{a+b}$

Students should be able to multiply any polynomial by a monomial. The teacher is urged to use common and decimal fractional values as the coefficients.

EXAMPLES:

$$\frac{2}{5}ab\left(\frac{5}{3}a + 10b - \frac{15}{7}c\right) = \frac{2}{3}a^2b + 4ab^2 - \frac{6}{7}abc$$

$0.6a(2a - 0.5) = 1.2a^2 - 0.3a$

Students should be able to multiply two binomials. Special attention should be given to the square of a binomial and to the product of conjugate binomials.

$(a + b)^2 = a^2 + 2ab + b^2$
$(a + b)(a - b) = a^2 - b^2$

EXAMPLES:

$(x + 2)(2x - 3) = 2x^2 + x - 6$
$(3x - 1)^2 = 9x^2 - 6x + 1$
$(2x + 3)(2x - 3) = 4x^2 - 9$

The multiplication of a binomial by a trinomial and the multiplication of two trinomials are optional.

Students should be able to add fractions with unlike denominators. Work with fractions having monomial denominators is required. Addition of fractions with denominators that are unlike polynomials (other than monomials) is optional.

EXAMPLE:

Find the sum of $\frac{x + 2}{5x}$ and $\frac{3x - 4}{10x}$ if $x \neq 0$.

$$\frac{x + 2}{5x} + \frac{3x - 4}{10x} = \frac{2}{2} \cdot \frac{x + 2}{5x} + \frac{3x - 4}{10x}$$

$$= \frac{2x + 4}{10x} + \frac{3x - 4}{10x}$$

$$= \frac{5x}{10x}$$

$$= \frac{1}{2} \text{ if } x \neq 0$$

Students should be able to apply the rules for multiplication to the multiplication of fractions with monomial denominators. Multiplication of fractions with polynomial (other than monomial) denominators is optional.

EXAMPLES:

$$\frac{a + 1}{3} \cdot \frac{a - 1}{2} = \frac{a^2 - 1}{6}$$

$$\frac{2}{3x} \cdot \frac{x - 1}{4} = \frac{x - 1}{6x}, \; x \neq 0$$

Simplification of Algebraic Expressions using Addition and Multiplication

Students should be able to identify operations necessary to simplify an algebraic expression, the order in which the operations are to be performed, and the techniques necessary to carry out the operations.

EXAMPLE:

$$5 - 3(x - 2) = 5 - 3x + 6$$
$$= 11 - 3x$$

Division of Polynomials by Monomials

Students should learn the rule for the division of powers with like bases by using it in the cases where $a > b$.

$x^a \div x^b = x^{a-b}$ if $x \neq 0$

It is then helpful to use this rule to justify the definitions of zero and negative exponents.

EXAMPLE:

$\frac{x^4}{x^4} = x^0$ if $x \neq 0$

Since $\frac{x^4}{x^4} = 1$, $x^0 = 1$ if $x \neq 0$

Similarly, $\frac{x^4}{x^6} = x^{-2}$ if $x \neq 0$

However, $\frac{x^4}{x^6} = \frac{1}{x^2}$ if $x \neq 0$

Therefore, $x^{-2} = \frac{1}{x^2}$ if $x \neq 0$

Examples such as these can lead the students to the definitions of zero and negative exponents.

If $x \neq 0$, $x^0 = 1$ and $x^{-a} = \frac{1}{x^a}$

Students should be able to use integral exponents to express decimal numbers in scientific notation. The use of calculators and computers has increased the need to understand and use scientific notation.

EXAMPLE:

$3470 = 3.47 \times 10^3$
$2.9 \times 10^{-6} = 0.0000029$

Students should be able to use the rules for the division of powers with like bases to simplify fractions with monomial denominators.

EXAMPLE:

$$\frac{5x^3 - 15x^2}{5x^2} = x - 3 \text{ if } x \neq 0$$

In all discussions of division, or of fractions, the exclusion of those values of the variables that make the divisor, or the denominator, equal to 0 should be noted.

Students should be able to apply the rules for division to the division of fractions with monomial denominators. When reducing fractions to lowest terms, students should be able to remove a common monomial factor from the numerator and denominator.

Factoring

Review, as needed, the terms prime and composite. Students should be able to factor a composite number into its prime factors.

EXAMPLE:

$180 = 2^2 \cdot 3^2 \cdot 5$

Before students can select the largest common factor of the terms of a polynomial, they must be able to identify all of the factors of each term.

EXAMPLE:

The factors of $3xy^2$ are 1, 3, x, y, 3x, 3y, xy, y^2, xy^2 3xy, $3y^2$, $3xy^2$.

Students should be able to find the largest monomial factor of a polynomial and to write the polynomial as the product of the monomial and a polynomial.

EXAMPLE:

$3x^3 - 9x = 3x(x^2 - 3)$

Students should be able to find the binomial factors of a trinomial, if they exist. Factoring should not be limited to those cases in which the leading coefficient is 1. Special attention should be given to the form of the perfect square trinomial which has two equal binomial factors, that is, the square of a binomial.

EXAMPLES:

$2x^2 - 7x + 6 = (2x - 3)(x - 2)$
$x^2 - 6x + 9 = (x - 3)(x - 3) = (x - 3)^2$

A binomial such as $x^2 - 16$ may be factored by using the methods employed for factoring a trinomial by writing it as $x^2 + 0x - 16$. However, factors of the difference of two squares represent such an important and useful case in mathematics that the students should readily recognize the binomial factors.

EXAMPLE:

$(x^2 - 16) = (x - 4)(x + 4)$

Finding three or more factors of a binomial or trinomial, if they exist, is optional.

EXAMPLE:

$$(x^4 - 16) = (x^2 - 4)(x^2 + 4)$$
$$= (x - 2)(x + 2)(x^2 + 4)$$

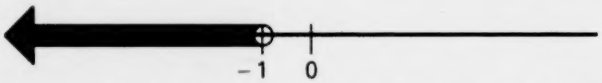
SOLUTION OF EQUATIONS AND INEQUALITIES

Open Sentences

Students have discussed the truth value of an open sentence in their study of logic. It is possible to list the solution set of open sentences such as $2x + 3 = -1$ and $x < 3$ when the domain is the set of integers. The solution set of $x < 3$ cannot be listed when the domain is the set of real numbers, and a graph is often used to express the solution set. Students should recognize how the graph of the solution set of the inequality $x < -1$ differs from that of the solution set of the inequality $x \leq -1$, which includes its boundary value.

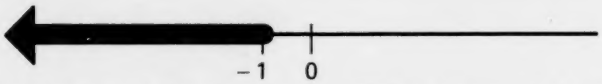
EXAMPLE:

Graph the solution set of $x < -1$ on a number line if the domain is the set of real numbers.



The open circle at -1 indicates that the boundary value, -1, is not included in the solution set.

Graph the solution set of $x \leq -1$ on a number line if the domain is the set of real numbers.



The closed circle at -1 indicates that the boundary value, -1, is included in the solution set.

Techniques for Solving Linear Equations and Inequalities

Review the use of inverse operations to solve equations of the form $x + a = b$, $ax = b$, and $ax + b = c$.

EXAMPLES:

$2x + 7$	$= 15$	$\frac{2}{3}x - 3$	$= 13$
$2x + 7 - 7$	$= 15 - 7$	$\frac{2}{3}x - 3 + 3$	$= 13 + 3$
$2x$	$= 8$	$\frac{2}{3}x$	$= 16$
$\frac{1}{2}(2x)$	$= \frac{1}{2}(8)$	$\frac{3}{2}\left(\frac{2x}{3}\right)$	$= \frac{3}{2}(16)$
x	$= 4$	x	$= 24$

The order in which the various steps are performed is frequently not unique. The solutions of these equations may also be written in set notation.

Skill in solving equations and inequalities should be developed gradually to include the solution of those involving parentheses and numerical coefficients expressed as common or decimal fractions. Students should be able to solve linear equations in which the variable appears on both sides and in which one or both sides must be simplified before solving the equation.

EXAMPLE:

$$x - 5(x + 4) = 2x - 6$$

Literal equations occur often in mathematics and the physical and social sciences. Students frequently experience literal equations when working with formulas.

EXAMPLE 1: Solve $ab + c = d$ for b.

Solution: $ab = d - c$

$$b = \frac{d - c}{a}$$

EXAMPLE 2: Solve $A = \frac{1}{2}bh$ for b.

Solution: $2A = bh$

$$\frac{2A}{h} = b$$

Ample time should be allowed for solving these types of equations.

Students should be able to use additive and multiplicative inverses to solve inequalities as well as equations. Multiplication or division of an inequality by a negative number requires special attention.

If $x > b$ and $a > 0$ then $ax > ab$
If $x > b$ and $a < 0$ then $ax < ab$
If $x < b$ and $a > 0$ then $ax < ab$
If $x < b$ and $a < 0$ then $ax > ab$

As students learn to solve linear equations and inequalities, they should apply these skills to the solution of problems expressed in words. Teachers should formulate problems based on number relationships, formulas, geometric concepts, and other areas in the curriculum. In addition to teaching problem-solving strategies and critical thinking skills, problems should be used to illustrate and reinforce the basic algebraic concepts and equations/inequalities presented in this course.

Problem Solving

1. The measures of the sides of a triangle are three consecutive odd integers. If the perimeter is 39 centimeters, find the measure of each side.

2. The length of a rectangular field is 20 meters longer than its width. Find the dimensions of the field if its perimeter is 120 meters.

3. Carol drove 180 kilometers in x hours, and then drove 210 kilometers in 2x hours. If the average speed for the total trip was 65 km/h, find x.

4. The atomic number of chlorine is 1 less than three times the atomic number of carbon. If twice the atomic number of carbon is subtracted from the atomic number of chlorine, the result is 5. Find the atomic number for each element.

5. John wants to earn at least $10 this week. His father agrees to pay him $4 for cutting the grass and $2.50 an hour for weeding the garden. If John cuts the lawn, what is the least number of hours he must spend weeding to earn at least $10?

6. In triangle ABC, the measure of $\angle B$ is twice the measure of $\angle A$ and the measure of $\angle C$ is 40 degrees more than the measure of $\angle A$. Find the measure of each angle.

7. At a sale a sweater sold for \$18. This amount was 80% of the original price. What was the original price?

8. A number increased by 20% of itself is 30. Find the number.

9. Janet spent 4 times as much as her sister Ann. If the girls spend \$24, how much did each girl spend?

10. Julie is 4 years older than Cathy. If Cathy's age is subtracted from three times Julie's age, the difference is 32 years. Find the age of each.

Techniques for Solving Quadratic Equations

From their study of arithmetic, students are familiar with the zero property of multiplication.

$$3 \cdot 0 = 0 \cdot 3 = 0 \text{ and } 0 \cdot 0 = 0$$

The product of two real numbers is 0 if and only if at least one of the factors is 0. This is the basic principle underlying the solution of a quadratic equation by factoring.

$$(ab = 0) \leftrightarrow [(a = 0) \vee (b = 0)]$$

Students often fail to relate this principle to the solution of a quadratic equation.

$2x^2 + x - 6 = 0$ if and only if one or both of the factors of $2x^2 + x - 6$ equals zero.

$$2x^2 + x - 6 = 0$$
$$(2x - 3)(x + 2) = 0$$
$$2x - 3 = 0 \text{ or } x + 2 = 0$$
$$x = \frac{3}{2} \quad \text{or} \quad x = -2$$

Solution of quadratic equations should be limited to those equations which can be solved by factoring over the set of integers. Students should be aware of the difference between the factors of a polynomial and the roots of a polynomial equation (the values of the variable that make the polynomial equation equal to 0).

Problems that can be solved using a quadratic equation can be drawn from the study of area, volume, number relationships, and science.

Problem Solving

1. The length of a rectangle is three meters more than twice the width. The area is 90 square meters. Find the dimensions.

2. Find two consecutive integers such that the square of the larger is 73 more than the smaller.

3. In a science experiment, Dale, who is standing on an observation tower 200 meters high, throws an object straight down toward the ground with an initial velocity of 30 m/sec. Using the distance formula, $s = \frac{1}{2}gt^2 + v_i t$ ($g = 10$ m/sec^2, v_i = initial velocity, and t = time), find the number of seconds required for the object to reach the ground.

4. In an orchard 160 trees are planted in rows, the same number of trees in each row. If the number of rows is 6 less than the number of trees in a row, find the number of rows.

5. The sum of the areas of two square plots is 130 square feet. The side of one plot is 2 feet longer than the side of the other. Find in feet the length of a side of each square.

6. Separate 18 into two parts whose product is 45.

7. Two numbers are in the ratio 2:3. Find the numbers if the square of the smaller number is 4 times the larger number.

8. One number is 5 more than a second number. Find the numbers if their product is 36.

The solution of a quadratic equation using the quadratic formula is optional and may be considered after the students have studied irrational numbers.

RATIO AND PROPORTION

Ratio

A ratio is a comparison of two numbers by division. The ratio $\frac{3}{4}$ may also be written as 3:4. The ratio is in simplest form when both terms are integers that have no factor other than 1 in common. If each term of a ratio is multiplied by the same nonzero number, the value of the ratio is unchanged. Thus, the ratio 3:4 is equivalent to 3x:4x for all x, $x \neq 0$.

EXAMPLE:

Two partners share $1000 in profits in the ratio 2:3. Find the share each partner receives.

Representing the shares by 2x and 3x leads to the equation

$2x + 3x = 1000$

Dimensional Analysis

In high school science courses, particularly chemistry and physics, students are often required to convert units of measure. Dimensional analysis is a technique used in such conversions.

In dimensional analysis, the unit of measure is treated as a factor in ratio form, e.g., $\frac{m}{sec}$ (meters per second), and, hence, can be "canceled." Although some mathematicians do not consider this process mathematically pure, it yields the desired unit of measure.

EXAMPLE 1:

Convert 90 km/hr. to m/min.

1 km = 1000 m
1 hr. = 60 min.

$$90 \frac{km}{hr.} \cdot \frac{1000\ m}{1\ km} \cdot \frac{1\ hr.}{60\ min} = 1500\ m/min.$$

EXAMPLE 2:

The distance a body falls is given by the formula $d = \frac{1}{2}gt^2$ where g is the acceleration due to gravity and t is the time of free fall. Find the distance a stone dropped from the top of a cliff will fall in 3 seconds (g = 9.8 meters per second per second).

$$d = \frac{1}{2} gt^2$$

$$d = \frac{1}{2} \times 9.8 \frac{m}{s^2} \cdot (3s)^2 = 4.9 \frac{m}{s^2} \cdot 9s^2 = 44.1\ m$$

This example of dimensional analysis provides an opportunity for the interdisciplinary teaching of mathematics and science.

Proportion

A proportion is an equation that expresses the equality of two ratios. The students should recognize a proportion written as $\frac{a}{b} = \frac{c}{d}$ or as a:b = c:d.

They should be able to identify a and d as the extremes and b and c as the means of the proportion.

Solution of Proportions

In any proportion, the product of the extremes equals the product of the means. An algebraic proof of this statement should be presented to the students.

$$\frac{a}{b} = \frac{c}{d} \qquad b \neq 0, d \neq 0$$

$$(bd)\frac{a}{b} = \frac{c}{d}(bd)$$

$$\frac{abd}{b} = \frac{bcd}{d}$$

$$ad = bc$$

Students should be able to solve proportions with monomial and binomial terms.

EXAMPLE 1:

$$\frac{x}{x-2} = \frac{6}{5} \qquad \text{for } x \neq 2$$

$$5x = 6x - 12$$

$$-x = -12$$

$$x = 12$$

EXAMPLE 2:

$$\frac{x}{4} = \frac{1}{x-3} \qquad \text{for } x \neq 3$$

$$x^2 - 3x = 4$$

$$x^2 - 3x - 4 = 0$$

$$(x-4)(x+1) = 0$$

$$x - 4 = 0 \text{ or } x + 1 = 0$$

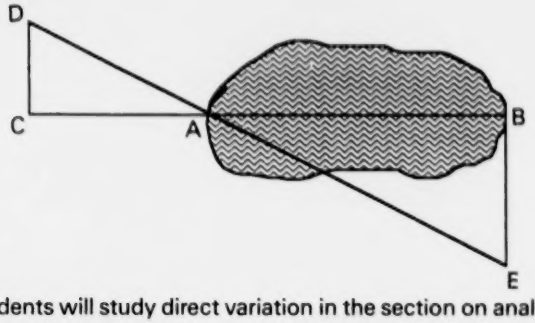
$$x = 4 \text{ or } x = -1$$

Applications

Percent should be recognized by students as the ratio of some number to 100. Therefore, problems involving percent can be solved using the same procedures as problems that use proportions in their solutions.

When students study congruent figures (See section on Geometry.), they will learn that if the angles of one triangle are congruent to the corresponding angles of a second triangle, the triangles are not necessarily congruent. They are, however, similar. As with congruence, it is suggested that similarity be dealt with on a more intuitive than formal level. The use of similar figures makes it possible to solve problems of indirect measurement.

EXAMPLE:



To measure the distance across a pond (AB), points C, D, and E are located so that $\overline{CD} \perp \overline{CA}$, $\overline{AB} \perp \overline{BE}$, AC = 10 meters, CD = 15 meters, and BE = 360 meters. Find the distance across the pond.

Students will study direct variation in the section on analytic geometry. Two quantities, x and y, vary directly if the ratio of any two corresponding values of these variables is constant.

$$\frac{x}{y} = k \text{ or } \frac{x_1}{y_1} = \frac{x_2}{y_2}$$

Everyday experiences present many situations in which two measurable quantities are directly proportional.

Problem Solving

1. The ratio of the width of a rectangle to its length is 1:5. Find the length and width of the rectangle if the length is 8 centimeters longer than the width.

2. A car is traveling at 20 m/sec. How fast is the car moving in km/hr.?

3. Using the formula M = DV where M = mass, D = density and V = volume, find the mass of an object whose volume is 2 cubic centimeters (2 cm^3) and whose density is 8 grams per cubic centimeter ($8 \frac{g}{cm^3}$)

4. The numerator of a fraction is 7 less than the denominator. If the numerator is decreased by 2 and the denominator is increased by 3, the value of the new fraction is $\frac{1}{3}$. Find the original fraction.

5. A solution of iodine and water is 3% iodine. How many mL of iodine are there in a 500 mL solution?

6. On a test, Ernestine answered 12 out of 15 questions correctly. What percent of the questions did she answer correctly?

7. From June to December, the price of a 5 pound bag of sugar increased from \$1.10 to \$1.32. What is the percent of increase during this time period?

8. The Apex Appliance Company advertises a color television set for \$509.15. This price represents a discount of 15% off the original price. What was the original price of the set?

9. The number of chirps made by a cricket varies directly as the temperature. If at 10 degrees Celsius the cricket chirps 40 times per minute, how many chirps per minute does he make at 15 degrees Celsius?

10. On a map, the scale of 2.5 cm represents 500 km. How many centimeters are used to represent 150 km? How many kilometers are represented by 10 cm?

11. A recipe calls for 2.5 cups of flour to make 90 cookies. How many cups of flour are needed to make 2 dozen cookies?

12. A meter stick casts a shadow 16 cm long. At the same time of day, a flagpole casts a shadow 2 m long. How tall is the flagpole?

REAL NUMBERS

Rational and Irrational Numbers

Every real number can be expressed as a finite or infinite decimal (or every real number can be expressed as an infinite decimal if finite decimals are considered to be infinite decimals with repeating 0, for example, $0.25 = 0.25\overline{0}$). Every real number is either rational or irrational.

A rational number is a number that can be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

EXAMPLES:

$$\frac{5}{9}, 4 = \frac{4}{1}, 3\frac{1}{2} = \frac{7}{2}, 0.6 = \frac{2}{3}, -\frac{3}{8} = \frac{-3}{8} = \frac{3}{-8}$$

Since a ratio is an indicated quotient, every rational number can be expressed as a decimal.

EXAMPLES:

$$5\overline{)3.0}\ 0.6 \text{ or } \frac{3}{5} = 0.6 = 0.6\overline{0} \qquad 6\overline{)5.00}\ 0.8\overline{3} \text{ or } \frac{5}{6} = 0.8\overline{3}$$

The decimal numerals that express a rational number either terminate (0.6) or repeat ($0.8\overline{3}$). A number can be expressed as a repeating decimal if and only if that number is rational.

If a real number is not rational, then it is called irrational and cannot be expressed as a repeating decimal. Therefore, an irrational number has a decimal representation that does not repeat.

Examples of Irrational Numbers

Since a non-repeating decimal is an irrational number, decimals such as 0.1234567891011121314...and 0.01001000100001000001... are irrational. Some of the irrational numbers that occur most frequently in mathematical calculations are expressed in radical form such as:

$$\sqrt[3]{3}, \qquad \sqrt[3]{4}, \qquad \sqrt{2}$$

If $x = \sqrt[n]{a}$, then $x^n = a$ or $\overbrace{x \cdot x \cdot x \cdot x \ldots}^{\text{n factors}} = a$. When no index, n, is written, it is understood to be 2. It should be noted that, although there are two numbers, one positive and one negative, each of whose square is 3, the symbol $\sqrt{3}$ stands for only the positive number. A proof by indirect reasoning that $\sqrt{3}$ is irrational may be presented as an optional exercise.

Another common irrational number is pi (π), the ratio of the circumference to the diameter of any circle.

Rational Approximation of Irrational Numbers

The use of calculators and computers has made the rational approximation of an irrational number readily available. However, students should be able to employ at least one of the common arithmetic methods for obtaining a rational approximation of an irrational square root. The divide and average method is useful because it can be readily programmed for a computer or calculator.

Operations with Radicals

The general rules $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ and $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ are used to simplify, to multiply, and to divide radicals. If ab is not a perfect square, then $\sqrt{ab}$ can be simplified if there is a factor of ab that is a perfect square. When ab is an integer that has no integral factor that is a perfect square, $\sqrt{ab}$ is in simplest form.

EXAMPLES:

$\sqrt{125} = \sqrt{25 \cdot 5} = 5\sqrt{5}$

$\sqrt{8 \cdot 2} = \sqrt{16} = 4$ or $\sqrt{8} \cdot \sqrt{2} = \sqrt{4}\sqrt{2}\sqrt{2} = 2 \cdot 2 = 4$

$\sqrt{15} \cdot \sqrt{5} = \sqrt{75} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$

Division may be employed before or after simplifying the terms of a fraction.

EXAMPLE:

$\frac{\sqrt{75}}{\sqrt{3}} = \sqrt{25} = 5$ or $\frac{5\sqrt{3}}{\sqrt{3}} = 5$

The distributive property is employed in writing the sum or difference of two like radicals as a single term just as it is in simplifying the sum or difference of two like algebraic terms.

EXAMPLES:

$2\sqrt{3} + 5\sqrt{3} = (2 + 5)\sqrt{3} = 7\sqrt{3}$

$3\sqrt{12} + \sqrt{27} = 3\sqrt{4} \cdot \sqrt{3} + \sqrt{9} \cdot \sqrt{3} = 6\sqrt{3} + 3\sqrt{3} = 9\sqrt{3}$

Students should be able to add, subtract, multiply, and divide two or more irrational numbers expressed as the square root of a positive integer or as the product of a rational number and the square root of a positive integer. Simplifying radicals with a fractional radicand should be limited to those in which the denominator of the fraction is a perfect square. Rationalizing the denominator of a fraction is optional.

Applications

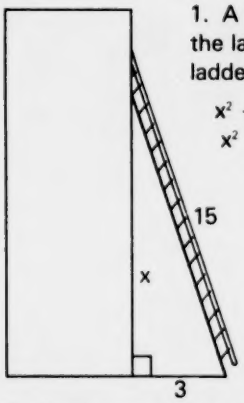
Quadratic equations of the form $ax^2 + bx + c = 0$ in which $b = 0$ can be solved without factoring.

EXAMPLE:

$$5x^2 - 40 = 0$$
$$5x^2 = 40$$
$$x^2 = 8$$
$$x = \pm\sqrt{8}$$
$$x = \pm 2\sqrt{2}$$

Note that there are always two square roots of a positive number. Facility with irrational numbers is important in the use of the Pythagorean theorem.

EXAMPLES:

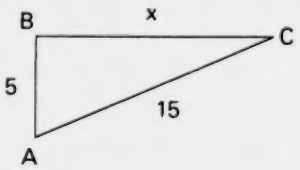


1. A ladder 15 feet long leans against a vertical wall. The foot of the ladder is 3 feet from the wall. How far up the wall does the ladder reach?

$$x^2 + 3^2 = 15^2$$
$$x^2 + 9 = 225$$
$$x^2 = 216$$
$$x = \pm\sqrt{216} = \pm\sqrt{36}\cdot\sqrt{6} = \pm 6\sqrt{6}$$

The ladder reaches a point that is $6\sqrt{6}$ feet up the wall. Notice that the negative root of the equation cannot be a solution to the problem and should be rejected.

2. From point A, Leonard travels 5 miles due north to point B, and then east to point C. If he is then 15 miles from his starting point A, how far east did he travel?



$$x^2 + 5^2 = 15^2$$
$$x^2 + 25 = 225$$
$$x^2 = 200$$
$$x = \pm\sqrt{200} = \pm\sqrt{100}\cdot\sqrt{2} = \pm 10\sqrt{2}$$
$10\sqrt{2}$ Ans.

He traveled $10\sqrt{2}$ miles to the east. The negative root of the equation cannot be a solution to the problem.

In this course, the derivation and use of the quadratic formula are optional.

Problem Solving

1. The diagonal of a rectangle is 12 cm. Find the length of the rectangle if its width is 4 cm.

2. The length of a rectangle is 3 times its width. If the diagonal of the rectangle is 15 feet, what are the length and width of the rectangle?

3. Jimmy and Tommy started from the same place. Jimmy traveled west at the rate of 30 mph and Tommy traveled south at the rate of 40 mph. How far apart were they at the end of 2 hours?

4. A baseball diamond is in the shape of a square, 90 feet on a side. What is the distance from home plate to second base?

5. Show that the diagonal of a square whose side is s can be found by the formula $d = s\sqrt{2}$.

6. Mr. Fitzgerald leaned a 36-foot ladder against a vertical wall of a building with one end placed 10 feet away from the wall on level ground. Find the distance from the top of the ladder to the ground.

III.

Geometry

Most of the geometric terms and figures that students will study in this section are already familiar to them. One of the goals of Course I is to apply the tools of algebra to familiar concepts in order to reinforce ideas and generalize about relationships. The teacher is encouraged to present, whenever possible, algebraic justification of assumptions about geometric figures. The topics of this section should not be taught as a self-contained unit, but should be introduced whenever another branch of mathematics can be used to demonstrate a geometric relationship or whenever geometry can be used to apply or reinforce a non-geometric concept. Suggestions for places where this can be done are indicated throughout each section.

Proof presented in Course I should be informal. The student should, at the end of Course I, be prepared to study more formal proof in Course II, with some familiarity with the theorems to be proven. This will enable the student to focus, in Course II, on the idea of proof rather than on the statements to be proven.

UNDEFINED TERMS

Point, Line, and Plane

Point, *line*, and *plane* are the usual undefined terms of geometry. From these intuitive ideas, definitions of other geometric constructs are formulated. In introducing these concepts, teachers should use symbolism carefully and encourage students to do the same. It is recommended that the symbolism and definitions used in Course I be those that the students will use in Course II.

REVIEW OF GEOMETRIC FIGURES

Parallel and Intersecting Lines

Two distinct lines can intersect in, at most, one point. In a plane, if two lines do not intersect, they are parallel.

Angles

Students should review the terminology and notation associated with angles.

a. degree measure (This is the only angle measure used in Course I.)

b. classification of angles by measure: acute, right, obtuse, and straight

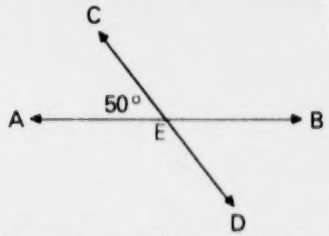
c. supplementary and complementary angles

d. vertical angles

Students should be able to recognize the pairs of supplementary angles and the pairs of vertical angles formed when two lines intersect.

EXAMPLE:

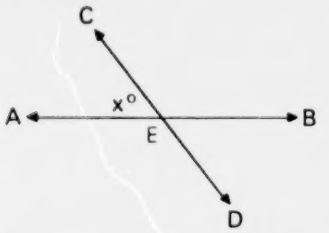
If m∠AEC = 50, find the measures of the other angles in the diagram.



m∠CEB = 130
m∠BED = 50
m∠DEA = 130

After examples like the one given above, students should observe that the vertical angles have the same measure. The teacher should present an algebraic argument similar to the following:

If m∠AEC = x degrees, the degree measures of the other angles in the diagram expressed in terms of x are:

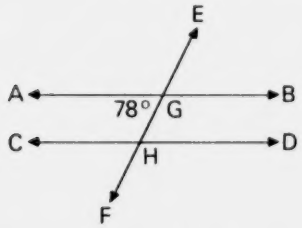


m∠CEB = 180 − x
m∠BED = 180 − (180 − x) = x
m∠DEA = 180 − x

Therefore, vertical angles AEC and BED have equal measures, x degrees, and vertical angles CEB and DEA have equal measures, (180 − x)°.

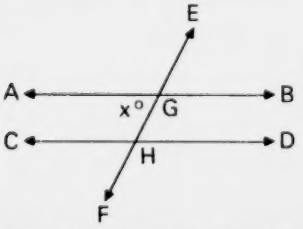
e. alternate interior, corresponding angles

When two lines are cut by a transversal, the alternate interior angles formed are congruent if and only if the lines are parallel. This statement can be accepted without proof.



If it is known that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and m∠AGH = 78, students should be able to find the measure of each angle in the diagram.

The use of numerical values can prepare the student to use variables to investigate the relationships of the measures of the angles formed when parallel lines are cut by a transversal.



If m∠AGH = x, the students may be asked to express all other angles in the diagram in terms of x. As a result of this exercise, corresponding angles and alternate exterior angles can be shown to be congruent, and interior angles on the same side of the transversal can be shown to be supplementary.

The following examples illustrate the use of these relationships in an algebraic setting.

EXAMPLE 1:

The degree measure of an angle is represented by $3x - 60$. Represent the degree measure of the complement of this angle. Represent the degree measure of the supplement of this angle. (Questions such as these should be used after students have learned to use variables to express numbers.)

EXAMPLE 2:

If $0 < m\angle A \leq 180$, and $m\angle A = 3x - 60$, what are the possible values of x?

If $\angle A$ is an acute angle and $m\angle A = 3x - 60$, what are the possible values of x?

If $\angle A$ is an obtuse angle and $m\angle A = 3x - 60$, what are the possible values of x?

Questions such as these may be used after students have learned to solve inequalities.

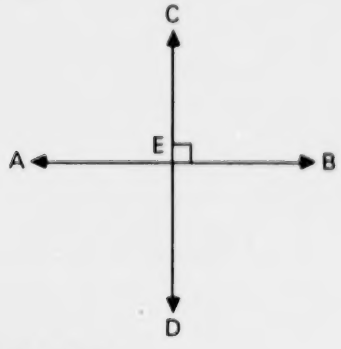
EXAMPLE 3:

The measure of an angle is 10 degrees more than the measure of its supplement. Find the measure of the angle and of its supplement. (Questions such as this can be used after students have begun to solve equations.) The complexity of such problems can be increased appropriately.

Perpendicular Lines

Two lines are perpendicular if and only if they intersect to form right angles.

EXAMPLE:



Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E. The degree measure of $\angle AEC$ is $2x + 30$. If lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are perpendicular, what is the value of x?

Simple Closed Curves

An informal definition of a *simple closed curve* should be reviewed and examples cited.

TRIANGLES

Classification

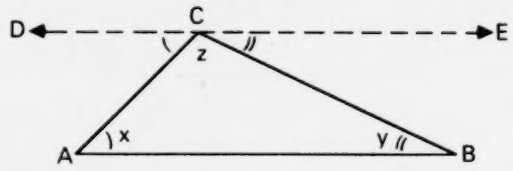
The classification of triangles should be reviewed:

- by sides – scalene, isosceles, equilateral
- by angles – acute, right, obtuse

Sum of the Measures of the Angles of a Triangle

An informal proof that the sum of the measures of the angles of a triangle is 180 degrees can be presented, making use of the students' knowledge of the measures of the alternate interior angles formed when two parallel lines are cut by a transversal.

EXAMPLE:

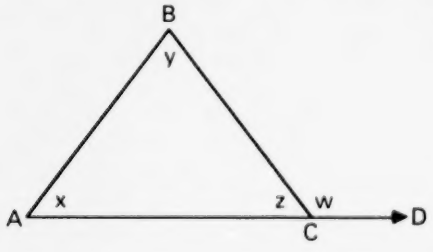


Triangle ABC is given with $m\angle A = x$, $m\angle B = y$, and $m\angle C = z$. Through C, draw $\overleftrightarrow{DE}$ parallel to $\overleftrightarrow{AB}$. Since $\angle A$ and $\angle DCA$ are alternate interior angles formed when the transversal $\overleftrightarrow{AC}$ intersects two parallel lines, these angles are congruent and have the same measure. Similarly, $\angle B$ and $\angle ECB$ are alternate interior angles formed when transversal $\overleftrightarrow{BC}$ intersects two parallel lines and have the same measure.

Therefore, $m\angle DCA = m\angle A$ or $m\angle DCA = x$
and, $m\angle ECB = m\angle B$ or $m\angle ECB = y$
Since $\angle DCE$ is a straight angle,
$m\angle DCA + m\angle ECB + m\angle ACB = 180$
Therefore, $x + y + z = 180$

Exterior Angle of a Triangle

The measure of an exterior angle of a triangle is equal to the sum of the measures of the two nonadjacent interior angles. An algebraic demonstration of this theorem can be presented after students have investigated many numerical examples.



Extend line segment $\overline{AC}$ to D forming an exterior angle, $\angle BCD$.

Let $m\angle A = x$
$m\angle B = y$
$m\angle ACB = z$
$m\angle BCD = w$
Then $x + y + z = 180$
or $x + y = 180 - z$
Also $w + z = 180$
or $w = 180 - z$
Therefore, $w = x + y$

Base Angles of an Isosceles Triangle

If two sides of a triangle are congruent, the angles opposite these sides are congruent. An informal demonstration of the truth of this statement could be presented by measuring the sides and angles of isosceles triangles with ruler and protractor or by an appeal to symmetry with respect to the altitude to the base. Students should be able to solve problems involving the measures of the angles of an isosceles triangle.

An alternative demonstration of this theorem can be presented after students have studied line reflections (See pg. 50.).

EXAMPLE:

The measure of the vertex angle of an isosceles triangle is 20 degrees less than the measure of a base angle. Find the measure of each angle of the triangle. Find the measure of an exterior angle at the vertex.

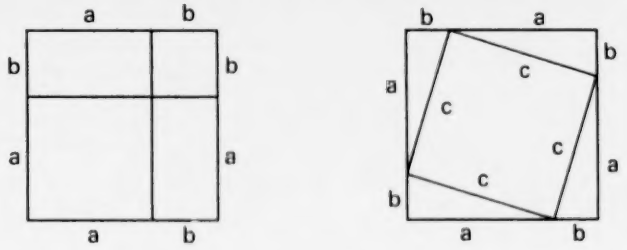
Comparison of Triangles

The concepts of congruence and similarity should be treated in an informal manner. Use sketches made carefully with protractor and ruler, or constructions with straightedge and compasses to investigate the conditions that are sufficient for congruence of triangles (SAS, ASA,

SSS, AAS) and those that are not sufficient (SSA, AAA). Work with similar triangles can be introduced when studying ratio and proportion. (See the section on Algebra.)

Pythagorean Theorem

Several informal proofs of the Pythagorean Theorem use procedures that involve only mathematics familiar to students at this level. The demonstration outlined below uses area formulas for a square and a triangle as well as the product of binomials.



$$(a + b)^2 = c^2 + 4(\frac{1}{2}ab)$$
$$a^2 + 2ab + b^2 = c^2 + 2ab$$
$$a^2 + b^2 = c^2$$

Another common proof uses the ratios of sides of the similar triangles formed when the altitude to the hypotenuse of a right triangle is drawn. Teachers may wish to present this proof to more able students.

QUADRILATERALS

Classification and Properties

Students should review the properties of common quadrilaterals:

Parallelogram
- Opposite sides are parallel
- Opposite sides are congruent
- Opposite angles are congruent
- Consecutive angles are supplementary

Rectangle
- All of the properties of a parallelogram
- All angles are right angles
- Diagonals are congruent

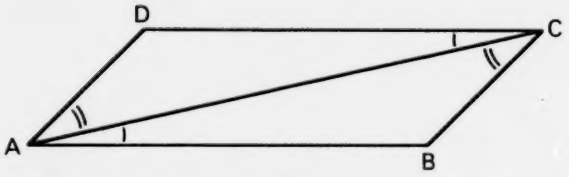
Rhombus
- All of the properties of a parallelogram
- All sides are congruent

Square
- All properties of a rectangle
- All properties of a rhombus

Trapezoid
- Exactly one pair of parallel sides

Many of these properties were investigated in earlier courses using measurement. Students may now be able to apply what they have learned about congruent angles formed when two parallel lines are cut by a transversal and the congruence of triangles to justify these statements about quadrilaterals.

EXAMPLE:



$\overline{AC}$ is a diagonal of parallelogram ABCD.

When $\overline{AB}$ and $\overline{DC}$ are cut by transversal $\overline{AC}$, $\angle DCA$ and $\angle BAC$ are congruent because they are alternate interior angles.

When $\overline{AD}$ and $\overline{BC}$ are cut by transversal $\overline{AC}$, $\angle DAC$ and $\angle BCA$ are congruent because they are alternate interior angles.

Side $\overline{AC}$ is a side of each of the triangles.

Therefore, triangles ABC and CDA are congruent by ASA.

The congruent corresponding sides and angles of these triangles are the opposite sides and angles of the parallelogram.

Sum of the Measures of the Angles

Since the diagonal of a convex quadrilateral separates the figure into two triangles, the sum of the measures of the angles of a quadrilateral (360°) can easily be obtained from the sum of the measures of the angles of the two triangles. This fact can also be verified for special quadrilaterals by considering the two pairs of supplementary angles in any quadrilateral with at least one pair of parallel sides.

POLYGONS

Review of Polygons

Students should review the names of the common polygons of more than 4 sides — pentagon, hexagon, octagon, and decagon.

Perimeter

Students should be familiar with the meaning of perimeter and the formulas for the perimeters of polygons, and should be able to solve problems involving perimeter (see the section on Algebra). The solution of a problem involving perimeter can often be an algebraic one.

EXAMPLE 1:

The measure of the base of an isosceles triangle is 3 cm less than twice the measure of one of the congruent sides. The perimeter of the triangle is 81 cm. Find the measure of each side of the triangle.

EXAMPLE 2:

The length of a rectangular photograph is 3 cm more than the width. The perimeter is 78 cm. Find the dimensions of the photograph.

This problem could be solved using one or two variables.

Let ℓ = length	Let w = width
w = width	w + 3 = length
$2\ell + 2w = 78$	$2w + 2(w + 3) = 78$
$\ell = w + 3$	

Formulas such as that for the perimeter of a rectangle provide useful applications of solving literal equations.

EXAMPLE:

Solve for ℓ:

$P = 2\ell + 2w$

$P - 2w = 2\ell$

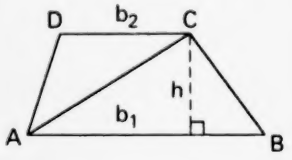
$\frac{P - 2w}{2} = \ell$

Area

Students should review informal derivations of formulas for the area of a parallelogram, a rectangle, a square, a triangle, and a trapezoid.

EXAMPLE:

Show that the formula for the area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$, where h is the height and b_1 and b_2 are the bases. The diagonal $\overline{AC}$ separates the trapezoid into two triangular regions.



Area of $\triangle ABC = \frac{1}{2}b_1h$

Area of $\triangle ADC = \frac{1}{2}b_2h$

Area of ABCD = area of $\triangle ABC$ + area of $\triangle ADC$

$= \frac{1}{2}b_1h + \frac{1}{2}b_2h$

$= \frac{1}{2}h(b_1 + b_2)$

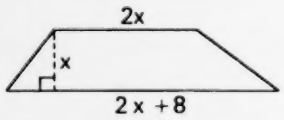
These formulas can make use of the students' skills in multiplying monomials and binomials, factoring, and solving quadratic equations.

EXAMPLE 1:

Find the height of a triangle whose base is 12 cm and whose area is 54 sq cm.

EXAMPLE 2:

The height of a trapezoid is represented by x. If one base is twice the height, and the second base is 8 more than the first base, represent the area of the trapezoid in terms of x.



$A = \frac{1}{2}x(2x + 2x + 8)$

$= \frac{1}{2}x(4x + 8)$

$= 2x^2 + 4x$

EXAMPLE 3:

Find the dimensions of a rectangular ice rink if the length is 10 meters more than twice the width and the area is 5500 square meters.

Let x = the width

$x(2x + 10) = 5500$

$2x^2 + 10x = 5500$

$2x^2 + 10x - 5500 = 0$

$x^2 + 5x - 2750 = 0$

$(x - 50)(x + 55) = 0$

$x - 50 = 0$ or $x + 55 = 0$

$x = 50$ or $x = -55$ (Reject this solution.)

The dimensions of the rink are 50 meters and 110 meters.

Students should understand that there are two solutions of the equation that was used to solve the problem, but only one of them is a member of the set of numbers (positive real numbers) that can be the measure of a side of a polygon.

EXAMPLE 4:

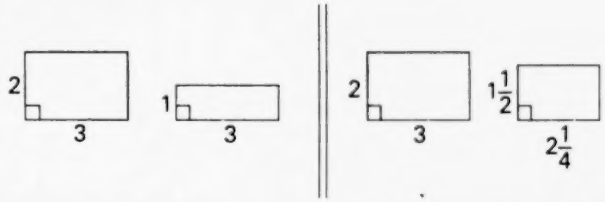
The measure of one leg of a right triangle is 2 cm more than twice the measure of the other leg. The area of the triangle is 12 sq cm. Find the measure of each leg.

Problems similar to examples 3 and 4 could be used after the students have learned to solve quadratic equations by factoring.

Similar Polygons

Triangles are similar if and only if their corresponding angles are congruent. Polygons of more than three sides are similar if and only if their corresponding angles are congruent and the measures of their corresponding sides are in proportion. Any two regular polygons of the same number of sides are similar.

EXAMPLES:

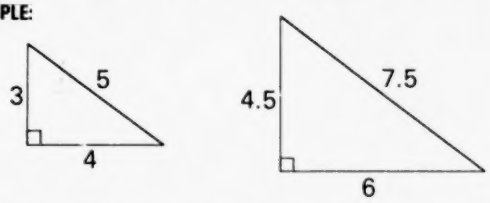


Corresponding angles are congruent.
Corresponding sides are not in proportion.
Figures are not similar.

Corresponding angles are congruent.
Corresponding sides are in proportion.
Figures are similar.

If two polygons are similar, the ratio of their perimeters is equal to the ratio of their sides, and the ratio of their areas is equal to the square of the ratio of their sides.

EXAMPLE:



Ratio of sides = 3:4.5 = 4:6 = 5:7.5 = 2:3
Ratio of perimeters = 12:18 = 2:3
Ratio of areas = 6:13.5 = 4:9 = $2^2:3^2$

CIRCLES

Circumference

Students should have experience measuring the circumferences and diameters of circles and should be able to calculate a reasonably accurate value of π by using the ratio of the circumference of a circle to the length of its diameter. The work with the circle should be done after students have studied irrational numbers, and they should be aware of the irrational nature of π. They should understand the difference between the exact value of the circumference expressed in terms of pi and the rational approximation obtained when a value such as 3.14 or 22/7 is used in place of π.

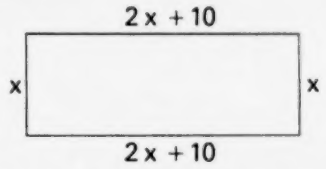
Area

Students should know and be able to use the formula for the area of a circle.

APPLICATIONS

The study of geometry is an inexhaustible source of problem material. Some "application" problems require the use of a geometric principle to write an equation that describes the given conditions. A diagram often helps in determining that equation.

EXAMPLE:



A rectangle whose length is 10 meters more than twice its width will require 200 meters of fencing to enclose it. What are the dimensions of the rectangle? What is the area of the rectangle?

Deductive problems that require the interaction of two or more geometric principles can be used to challenge students to use creative ways to solve problems.

EXAMPLE:

What is the rectangle of largest area that can be formed if the side of a barn is one side of the rectangle and 200 meters of fencing form the other three sides?

Problem Solving

1. One angle of a triangle is 12 degrees larger than a second angle. The third angle is 20 degrees less than the sum of the other two. Find the number of degrees in each angle of the triangle.

2. Two complementary angles are in the ratio 1:8. Find the number of degrees in the smaller angle.

3. Find the measure of each angle of an isosceles triangle if the vertex angle is 30° more than each base angle.

4. The measure of the supplement of an angle A is 70° more than 3 times the measure of its complement. Find the measure of angle A.

5. Two parallel lines are cut by a transversal so that two interior angles on the same side of the transversal are in the ratio 4:5. Find the number of degrees in each angle.

6. A person 6 feet tall casts a shadow 15 feet long. At the same time, a nearby tower casts a shadow 100 feet long. Find the number of feet in the height of the tower.

7. A rectangular room is 15 feet by 18 feet. How long, to the nearest tenth of a foot, is a diagonal of the room?

8. The length of a rectangle is 3 feet less than its width. If the perimeter of the rectangle is 42 feet, what is its area?

9. The sides of a triangle are in the ratio 3:4:5. Find the measure of each side if its perimeter is 42 centimeters.

10. An edge of a cube measures 4 cm. Find the total area of the surface of the cube.

11. A utility pole is supported by a guy wire 35 feet long. The guy wire is anchored to the ground at a point 28 feet from the foot of the pole. How high up the pole is the wire attached?

12. The measure of each exterior angle of a regular polygon is twice the measure of each interior angle. How many sides does the polygon have?

13. In triangle RST, the measure of angle R is four times the measure of angle S. An exterior angle at T measures 125 degrees. Find the measure of angle R.

14. The angles of a quadrilateral are in the ratio 3:4:5:6. Find the number of degrees contained in the largest angle of the quadrilateral.

15. The length of a rectangle is 1 centimeter less than twice its width. If the perimeter of this rectangle is 76 centimeters, find the width and the length.

16. The complement of an angle is 15° less than twice the angle. Find the measure of the angle.

17. A square and a rectangle have the same area. The length of the rectangle is 8 centimeters more than a side of the square, and the width of the rectangle is 4 centimeters less than a side of the square. Find the length of a side of the square.

TRANSFORMATIONS

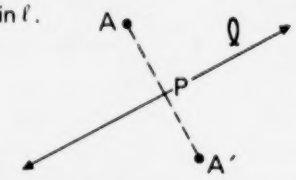
Intuitive Notions

As students study geometric figures, they should be directed to think informally about the ideas associated with reflections, translations, rotations, and dilations. Students can experiment with physical objects and/or sketch the image of a transformation using rulers, protractors, and models of a right angle (corner of a square).

A *reflection* can be considered the "mirror image" of a set of points.

EXAMPLE 1:

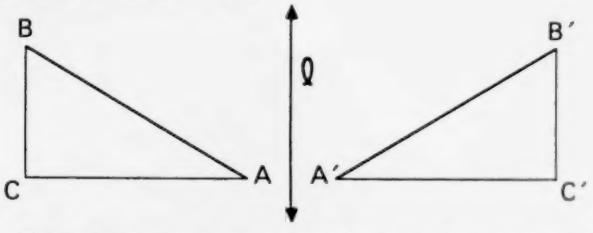
Sketch the reflection of A in ℓ.



Procedure: By using a protractor or a "square corner," draw a line through A perpendicular to and intersecting ℓ at P. Using a ruler extend AP its own length to A′. A′ is the reflection of A in ℓ.

EXAMPLE 2:

Sketch △ABC's reflection in line ℓ.

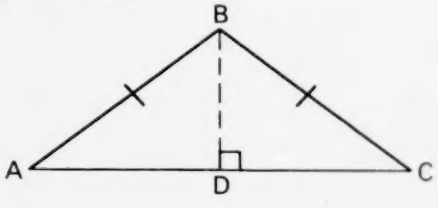


Procedure: This reflection can be accomplished by reflecting A, B, C, using the procedure shown in Example 1.

The "Base Angles of an Isosceles Triangle" theorem (See pg. 44.) can be demonstrated using line reflections.

EXAMPLE:

Given △ABC with $\overline{AB} \cong \overline{BC}$, prove that $\angle A \cong \angle C$.

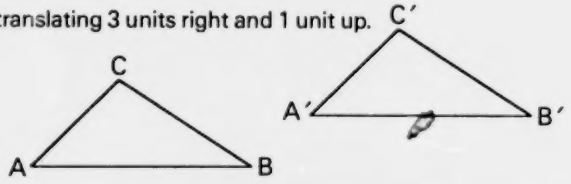


Procedure: By sketching the altitude from B to $\overline{AC}$, it can be shown, by measuring with a ruler or by paper folding, that $\triangle BDC$ is the mirror image of $\triangle BDA$ in $\overline{BD}$. Hence $\angle A \cong \angle C$.

A *translation* can be considered a "slide" of a figure in a plane.

EXAMPLE:

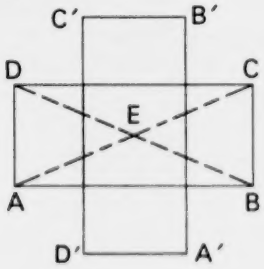
Sketch $\triangle ABC$ after translating 3 units right and 1 unit up.



Procedure: 1. Using a ruler or any unit length, measure 3 units to the right and 1 unit up for each vertex of the triangle ABC.
2. Using a ruler, draw $\overline{A'B'}$, $\overline{B'C'}$ and $\overline{C'A'}$, yielding $\triangle A'B'C'$.

A *rotation* can be thought of as rotating or turning a figure about some fixed point in the plane.

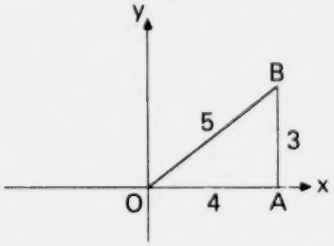
EXAMPLE 1:



Rotate rectangle ABCD 90° counterclockwise about point E.

EXAMPLE 2:

After reviewing the Rectangular Coordinate System (See pg. 55.), students could discuss the following informally.

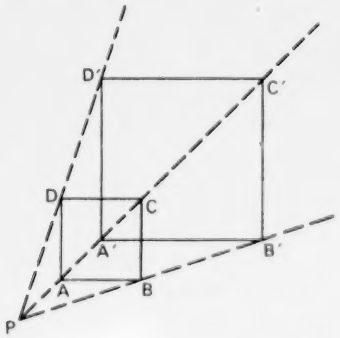


Rotate triangle OAB 90°, 180°, 270°, and 360° about point A. Discuss the effect of the rotation upon the sides of the triangle.

Students or teacher might do this activity using a physical model.

A *dilation* can be thought of as an enlargement or reduction of a figure with a fixed point of reference.

EXAMPLE:



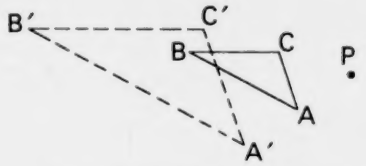
Dilate square ABCD so that the length of its side is twice as long as the side of the original square.

Congruence and Similarity

Students should recognize that reflections, translations, and rotations of objects preserve lengths of sides and measures of angles (size and shape) and thus produce figures congruent to the original. However, a dilation preserves only angle measure, not the length of sides. A figure formed by a dilation is similar (same shape, but not size) to the original.

EXAMPLE:

Sketch a triangle A′B′C′ similar to given triangle ABC with A′B′ = 2 AB.



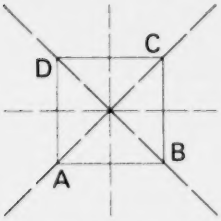
Procedure:
1. Choose any point P as the center of dilation.
2. Using the ruler with zero at P and passing through A (B and C), "construct" A′ (B′ and C′) to be the point on the ruler at twice the value of A (B and C).

Teachers should make use of the transformation concepts presented in this section when presenting the material in the analytic section of the syllabus.

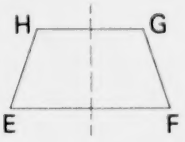
Symmetry

Students should be able to identify shapes which exhibit line and/or point symmetry. A figure has line symmetry if a line can be drawn through the figure such that one part of the figure is the "mirror image" of the other part in this line.

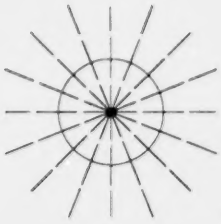
Examples of figures with line symmetry:



4 lines of symmetry



1 line of symmetry

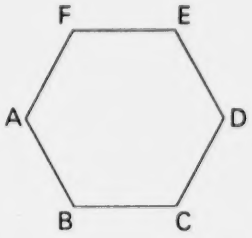


An infinite number of lines of symmetry

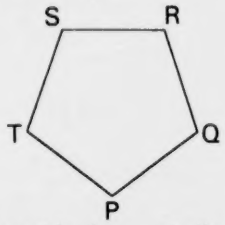
Example of a figure with no line symmetry:

A figure has point symmetry if the figure coincides with itself when rotated in either direction 180° about some fixed point.

Examples of figures with point symmetry:



Example of a figure with no point symmetry:



In addition to geometric shapes studied in this course, students can find many examples of symmetry all around them. Leaves, butterflies, people's faces, some letters of our alphabet, and well known pieces of art and architecture all exhibit symmetry.

SOLIDS

Classification

Students should be able to identify objects whose models are the common solids — prism, rectangular solid, pyramid, right circular cylinder, cone, and sphere.

Volume

Students should understand the meaning of volume. The prism and the cylinder have two congruent bases, and their volumes can be found by using the formula $V = Bh$ where B is the area of a base and h is the height of the solid figure. The pyramid and the cone have only one base, and their volumes can be found by using the formula $V = \frac{1}{3}Bh$. Students should be able to make use of these formulas in problem solving.

EXAMPLE:

An ice cream company wants to sell a product in the shape of a right circular cone. The diameter of the base of the cone is to measure 8 cm, and the cone will hold 250 cm^3 of ice cream. What should be the height of the cone to the nearest tenth of a centimeter?

Comparison of Volumes of Similar Solids

If the ratio of corresponding linear dimensions of two similar solids is a:b, then the ratio of the volumes of the similar solids is $a^3:b^3$.

EXAMPLE:

A laundry detergent is sold in two sizes, with boxes in the shape of similar rectangular solids. The height of the smaller box is 12 cm, and the height of the larger box is 18 cm. If the smaller box holds 16 ounces of detergent, what is the weight of the contents of the larger box?

IV.

Analytic Geometry

The development of a "graphing sense" is an important goal of Course I. Much of the future development of Courses II and III will rely on this "graphing sense". The students' work in 7th and 8th grade introduced them to the rectangular coordinate system as well as graphing on a number line. They are now ready to explore the connection between a linear algebraic sentence (equality or inequality) and its graphical representation on the coordinate plane. This unit is primarily a study of linear relations (first degree). Systems of linear equations or inequalities are approached both from an analytical and an algebraic point of view.

REVIEW OF RECTANGULAR COORDINATE SYSTEM

Students should review the terminology and conventions associated with the rectangular coordinate system.

- Coordinate axes (x-axis and y-axis)
- Origin
- Quadrants
- Coordinates of a point (ordered pairs):
  abscissa (x-coordinate), ordinate (y-coordinate)
- General representation of a point (x,y)

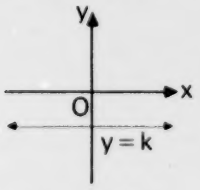
LINEAR RELATIONS AND SLOPE

Graphing Linear Relations

Linear relations such as $x + y = 5$, $x - 2y = 6$, $x = 8$, and $y = -2$ should be fully explored. Students should develop an understanding that there is an infinite number of ordered pairs (x,y) which algebraically satisfy each equation. They should have ample practice in graphing these relations on a coordinate system by plotting ordered pairs belonging to the relation. Using translations, investigate related graphs:

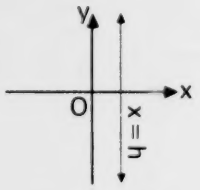
a) $y = x, y = x + 2, y = x - 3 \ldots y = x + a$
b) $y = ax, y = ax + 2, y = ax - 3 \ldots y = ax + b$

This graphing experience should lead students to see that such linear relations fall into the following categories:



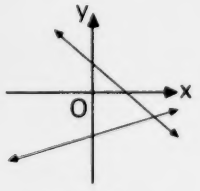
horizontal line

$y = k$



vertical line

$x = h$



oblique lines

$ax + by = c$

$(a \neq 0, b \neq 0)$

The graph of the linear relation $ax + by = c$ is a horizontal line if $a = 0$ and $b \neq 0$, is a vertical line if $b = 0$ and $a \neq 0$, and is an oblique line if $a \neq 0$ and $b \neq 0$.

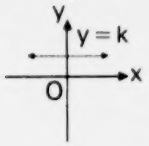
Slope of a Line

The concept of slope should be developed from the students' experience with graphing linear relations. Given any two points on a line, the slope of that line is intuitively thought of as the *rise* compared with the *run* associated with moving from one of these points to the other point. More formally, the slope, m, is defined as $\frac{\Delta y}{\Delta x}$ (the change in y compared to the change in x), or is expressed by the formula:

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

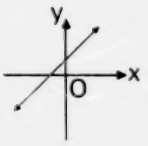
Students should understand that $\frac{y_1 - y_2}{x_1 - x_2}$ is equivalent to $\frac{y_2 - y_1}{x_2 - x_1}$ and that the choice of points is immaterial.

Students should understand how the value of the slope of a line identifies whether the line is horizontal, vertical, or oblique.

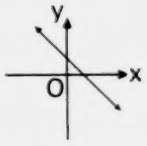


$m = 0$
horizontal line

m is undefined
vertical line



$m > 0$



$m < 0$

oblique lines

Students should understand the difference between a slope of 0 and a slope which is undefined.

Slope Intercept Form

The slope intercept form ($y = mx + b$) for graphing lines should be developed. Students should be given ample practice solving $ax + by = c$ algebraically for y to determine the slope and the y-intercept. After identifying the values of m and b, students should use these to graph the function. They should be aware of the importance of the sign of the slope, and should develop the "sense" that graphs of lines with positive slopes will rise from left to right and that the graphs of lines with negative slopes will fall from left to right.

DIRECT VARIATION

Problems involving two variables which vary directly with each other will have been studied in the ratio and proportion sections of the Algebra unit. Two quantities vary directly with one another if the ratio of any two corresponding values of these variables is constant, that is, $\frac{y}{x} = k$. Another form of this equation is $y = kx$. It should be noted that the graph of $y = kx$ is an oblique line with y-intercept of 0 and slope of k.

Many examples of direct variation can be chosen from the students' experiences in mathematics and science as well as from everyday experiences. Examples of direct variation include:

a) The perimeter of a square varies directly as the length of a side.

b) The distance traveled at a constant speed varies directly as the time traveled.

c) The amount of money earned in an 8-hour day varies directly as the hourly wage.

Students can experiment with ideas of direct variation such as those above by constructing tables of ordered pairs satisfying the statement of the variation. The following tables contain a few ordered pairs satisfying each of the above statements.

a) Let p = perimeter of the square

s(cm)	p(cm)
1	4
2	8
5	20
8	32

b) If rate = 50 mph

t(hrs)	d(mi.)
1	50
1 1/2	75
2	100
3	150

c) Let p = money earned in an 8-hour day

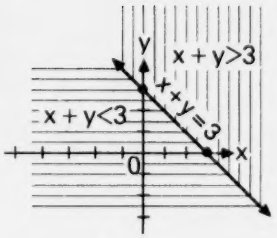
r($)	p($)
3.65	29.20
4.00	32.00
5.00	40.00
8.00	64.00

Each of these sets of ordered pairs can be graphed demonstrating that the graph of a direct variation is a linear function. As one variable increases, so does the other, proportionally. In the equations $\frac{y}{x} = k$ or $y = kx$, the value of k is the constant of proportionality. The student should be reminded that this constant value k is the slope of the linear function.

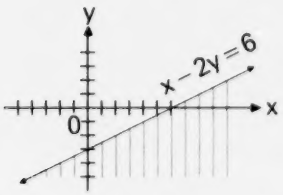
The topic of direct variation and its linear representation offers a variety of ways of presenting problems to students. Examples include:

1) If a 25-acre field of corn produces 3000 bushels of grain, how much will a 40-acre field produce at the same rate?
2) Draw a graph of a linear relation showing that the perimeter of an equilateral triangle varies directly as the length of one side.
3) If the interest (I) on $1000 varies directly as the rate (r), what change takes place in I when r is
   a) doubled?
   b) halved?
   c) multiplied by h?

GRAPHING LINEAR INEQUALITIES



The graph of $ax + by = c$ divides the plane into three sets of points, points on the line for which $ax + by = c$, points on one side of the line for which $ax + by < c$, and points on the other side of the line for which $ax + by > c$. For example, the graph of $x + y = 3$ separates the plane into a line and two open half planes. All ordered pairs on one side of the line satisfy $x + y < 3$, and ordered pairs on the other side of the line satisfy $x + y > 3$.



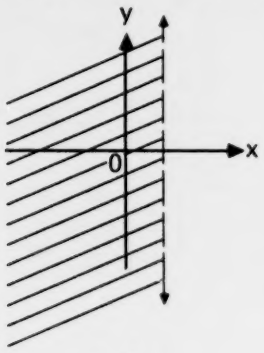
An inequality such as $x - 2y \geq 6$ ($x - 2y$ is greater than 6 OR $x - 2y = 6$) is represented by the union of two sets of points:

a) those on the line such that $x - 2y = 6$ or $y = \frac{1}{2}x - 3$

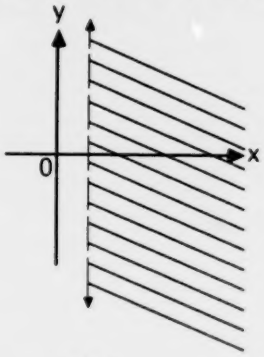
b) those in the half-plane below the line such that $x - 2y > 6$ or $y < \frac{1}{2}x - 3$

Note: The usual convention for graphing an inequality is to dot the line if it is not to be included and to make the line solid if it is included.

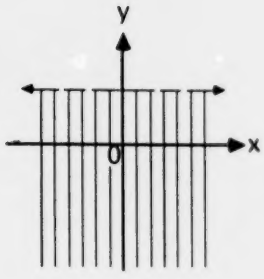
Students should work with inequalities of the type $x < k$, $x > k$, $y < h$, or $y > h$, in which the line separating the plane into two half-planes is either a vertical or a horizontal one.



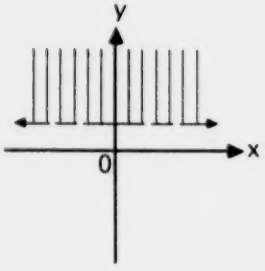
$x < k$



$x > k$



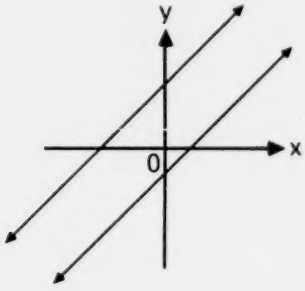
$y < h$

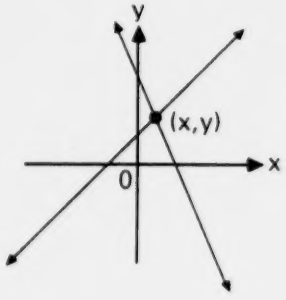


$y > h$

GRAPHIC SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

If two distinct lines are sketched on the same set of axes, they will either be parallel or will intersect in exactly one point.





It should be noted that the graphs of equations such as $x + 2y = 3$ and $3x + 6y = 9$ are coincident lines (the same line) since the two equations are equivalent.

Students should understand that if two distinct lines intersect, there is only one ordered pair (x,y), the coordinates of the point of intersection, which satisfies both relations simultaneously. This ordered pair is known as the solution of the system of equations.

Students should check that the ordered pair (x,y), determined from their graph as the solution, does in fact satisfy both linear relations.

ALGEBRAIC SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

Often the use of a graph to solve a system of equations is cumbersome and/or the solution of the system is difficult to determine accurately. The students should be introduced to two common algebraic methods of solution — the substitution method and the addition-subtraction method. Each of these methods eliminates one of the variables and reduces the system to one equation in one unknown.

Substitution Method

Solve the system:

$$2x - y = 5$$
$$x + 4y = 7$$

From the second equation, we obtain $x = 7 - 4y$

So, $2(7 - 4y) - y = 5$

or $14 - 8y - y = 5$

$14 - 9y = 5$

$-9y = -9$

$y = 1$

and $x = 7 - 4(1)$

or $x = 3$

Thus, the solution is (3,1).

Addition-Subtraction Method

$2x - y = 5$ $\quad - - -$ $8x - 4y = 20$

$x + 4y = 7$ $\quad - - -$ $x + 4y = 7$

Adding, $9x = 27$

$x = 3$

and $y = 1$

Thus, the solution is (3,1).

Application

A pair of linear equations in two variables can be used to solve problems by using any one of the preceding methods. Students should be encouraged to estimate answers to the problems prior to finding their solutions.

Example 1:

The sum of two numbers is 17, and their difference is 3. Find the numbers.

Let x = one number

y = the other number

$x + y = 17$

$x - y = 3$

Example 2:

One hundred tickets are sold for a dance. Some are couples tickets selling for $5 a couple. Others are singles tickets selling for $3 each. If the sales total $420, how many of each type of ticket were sold?

Let c = the number of couples tickets

Let s = the number of singles tickets

$c + s = 100$

$5c + 3s = 420$

Problem Solving

1. The sum of two numbers is 36. Their difference is 24. Find the numbers.

2. The owner of a men's clothing store bought 6 shirts and 8 hats for $140. One week later, at the same prices, he bought 9 shirts and 6 hats for $132. Find the price he paid for a shirt and the price he paid for a hat.

3. Four baskets of strawberries and 3 baskets of raspberries cost $11. At the same unit prices, two baskets of strawberries and 2 baskets of raspberries cost $6.50. Find the cost of one basket of each.

4. The length of a rectangular lot is 36 feet more than twice its width. Find the length and width of the rectangular lot if its perimeter is 504 feet.

5. Franz has 15 coins, some nickels and some dimes. The value of his coins is $1.25. How many of each kind does he have?

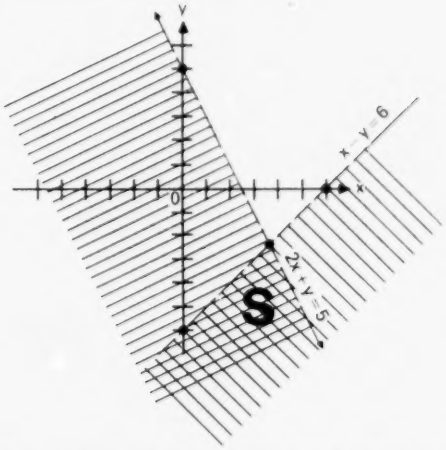
6. The Junior class members at Academia High School put on a class play. They sold 600 tickets, some student tickets and some adult tickets. The cost of a student ticket was $2, and the cost of an adult ticket was $3.50. How many of each kind of ticket were sold if the total receipts were $1800?

GRAPHIC SOLUTION OF SYSTEMS OF LINEAR INEQUALITIES

The solution set of a system of linear inequalities is the region of the coordinate plane which is common to both inequalities, i.e., the intersection of the two shaded areas.

Example:

Graph the solution of $2x + y < 5$ and $x - y > 6$.



S represents the infinite set of points (ordered pairs) which satisfy both inequalities simultaneously.

Students should be able to state comparable conditions for which each of the other regions is a solution set.

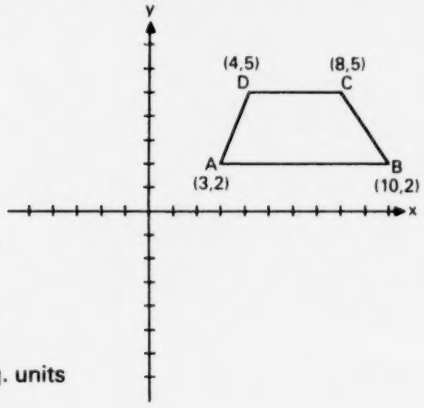
AREA USING COORDINATES

Students should apply the formulas for area of squares, rectangles, parallelograms, triangles, and trapezoids to problems stated in terms of the coordinates of the vertices of a figure. If bases and/or heights of the figure fall on horizontal or vertical lines of the graph paper, the lengths can be determined by counting. Otherwise, the figure may be "divided up" into simpler shapes so that the common area formulas can be used.

Example 1:

Find the area of trapezoid ABCD if the coordinates of its vertices are A(3,2), B(10,2), C(8,5), and D(4,5)

By sketching and counting, $b_1 = 7$, $b_2 = 4$, and $h = 3$.



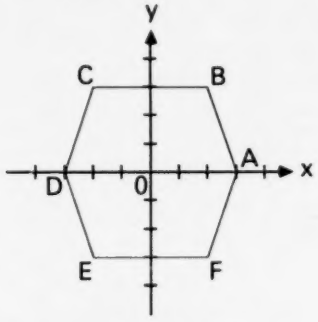
Using $A = \frac{1}{2}h(b_1 + b_2)$

$A = \frac{1}{2}(3)(7 + 4)$

$A = \frac{33}{2}$ or $16\frac{1}{2}$ sq. units

Example 2:

Find the area of hexagon ABCDEF if the coordinates of its vertices are A(3,0), B(2,3), C(−2,3), D(−3,0), E(−2,−3), F(2,−3).

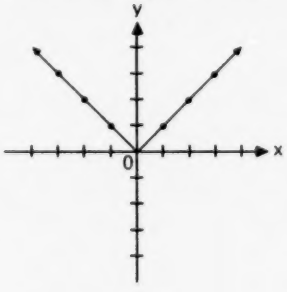


After sketching, several ways of "cutting up" the figure can be seen:

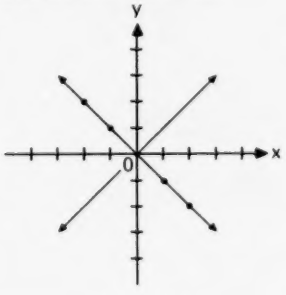
1) two trapezoids (ABCD and DEFA)
2) two triangles and a rectangle (△ABF, △CDE, and rectangle BCEF)
3) four triangles (△ABC, △CDA, △ADE, △AEF)

OPTIONAL GRAPHING

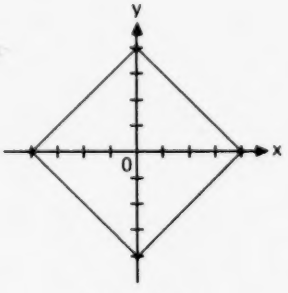
If optional graphing activities are desired, equations involving absolute value of one or both variables provide interesting graphs for students.



$y = |x|$



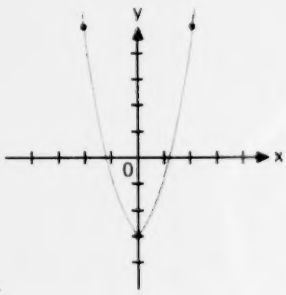
$|y| = |x|$



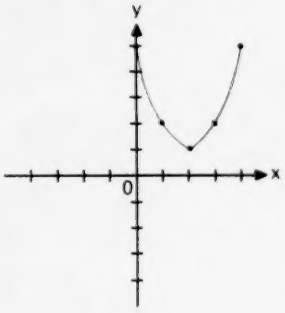
$|x| + |y| = 4$

The parabola is an important part of the coordinate work in Course II. Students in Course I may optionally study this curve by graphing $y = ax^2$, $y = ax^2 + b$, and $y = ax^2 + bx + c$.

$y = x^2$



$y = 2x^2 - 3$



$y = x^2 - 4x + 5$

V.

Probability

Probability is a common strand of the mathematics program in K-6 and 7-8. The students have been exposed to many of the basic notions of probability in an intuitive way. Teachers of Course I will need to assess the degree and depth of understanding of probability topics previously taught and use the material in this section per the background of the students. The emphasis of this probability section should be on applying the notions and techniques of probability theory to the analysis and solution of problems. For the more capable students, teachers may employ the notions of set theory in their discussion of the topics presented below.

Probability is an area in the curriculum in which teachers should incorporate estimation activities on a regular and sustaining basis. Students should be encouraged to predict and assess what a reasonable answer may be, particularly when determining a probability empirically.

REVIEW OF TERMS

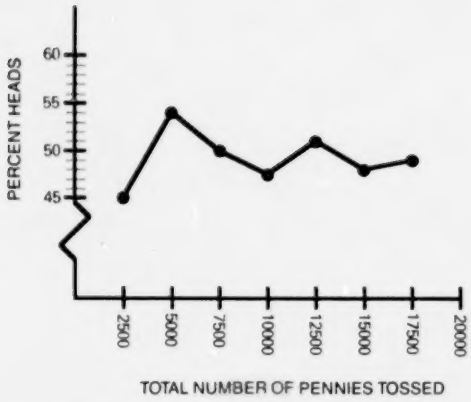
- probability of an event
- sample space
- outcomes
- successes (favorable outcomes)
- failures (unfavorable outcomes)
- equally likely events
- fair
- biased
- random

PROBABILITY OF SINGLE EVENTS

Theoretical Versus Empirical Probability

Students should develop an appreciation of the difference between theoretical probability and empirical probability. The prediction of the probability of getting a head upon tossing a fair coin or of getting a "six" upon tossing a fair six-faced die is $\frac{1}{2}$ or $\frac{1}{6}$, respectively, illustrating the theoretical, or ideal probability of these events. Students should spend ample time experimenting with probability devices such as spinners, coins, and dice to empirically determine probability (by observation) based on a large number of actual trials.

A suggested experiment is to have each member of the class toss 10 coins 10 times each night and to report the results the next day. A running account of the percent of heads should be kept and represented in a graph such as the following:



As the days go by and the total number of trials increases, the students will see on the graph a stabilization of the relative frequency toward the ideal or theoretical value. They need to understand that probability values determined empirically need to be based on a large number of trials.

In many cases one cannot determine theoretically a probability value, and can only approximate one empirically. Students should perform experiments to determine the probability of an event empirically. A suggested experiment is to throw a toothpaste cap a large number of times and to record the number of times each of the following positions results.

Side Bottom Top

In the statistics section, students will see how the compilation of recorded data can lead to establishing a probability value empirically. Consider a life insurance company's statement concerning life expectancy: "The probability of a U.S. male living beyond 75 years is $\frac{42}{100}$."

This means that the accumulated statistics show that for every group of 100 males born 75 years ago, about 42 are still living.

Simple Probability and Notation

Students should use the proper notation in expressing the probability of an event:

$$P(E) = \frac{\text{number of successes}}{\text{number of total outcomes}}$$

Do not make an issue of reducing fractions since such reductions actually obscure the significance of the values in the ratios.

The probability of an event is frequently expressed as a percent. Students are familiar with statements such as, "There is an 80% chance of rain tomorrow."

It should be noted that $0 \leq P(E) \leq 1$, since the numerator of the probability ratio represents the number of favorable outcomes. This number cannot be greater than the total number of possibilities. $P(E) = 1$ means E is certain to occur, and $P(E) = 0$ means E is impossible.

EXAMPLE 1:

What is the probability of rolling an odd number in one roll of a fair die?

Solution:

Since there are 3 "successes," (1, 3, 5), P (odd) $= \frac{3}{6}$

EXAMPLE 2:

What is the probability of drawing an ace from a standard deck of 52 cards?

Solution:

Since there are 4 aces in the deck, P(ace) = $\frac{4}{52}$

EXAMPLE 3:

What is the probability of obtaining a sum of 1 when two dice are rolled?

Solution:

Since this outcome is impossible, the probability is 0.

Problems Involving AND

The use of AND in probability problems can be related to the use of AND in logic and INTERSECTION in set theory.

EXAMPLE 1:

A card is drawn from a standard deck of cards. What is the probability that it will be both a seven and red?

Solution:

In order to get both a seven and a red card, the card must belong to both the set of sevens and the set of red cards (or, in other words, belong to the intersection of the two sets). Thus, the favorable outcomes include only the seven of hearts and the seven of diamonds. Thus, P(E) = $\frac{2}{52}$

EXAMPLE 2:

One number is selected from the integers from 1 to 10, inclusive. What is the probability that the number is an odd number and a prime number?

Solution:

The odd numbers are 1, 3, 5, 7, 9.
The prime numbers are 2, 3, 5, 7.
The numbers which are odd and prime are 3, 5, 7.
Thus, P(E) = $\frac{3}{10}$

Problems Involving OR

The use of OR in probability can be related to the use of OR in logic and UNION in set theory.

Care must be taken in problems dealing with OR since if an outcome is included in each of two or more sets, it should not be counted more than once.

EXAMPLE 1:

What is the probability of drawing a seven or a red card from a standard deck?

Solution:

There are 4 sevens and 26 red cards. However, 2 of the sevens are also red cards. Hence, the number of favorable outcomes is 4 + 26 − 2. Therefore, P(E) = $\frac{28}{52}$

EXAMPLE 2:

What is the probability of drawing a red ace or a black queen from a standard deck of cards?

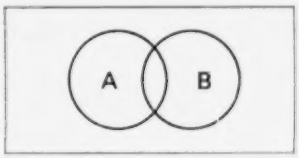
Solution:

In this problem there is no "overlap", and the number of favorable outcomes is simply found by adding the number of red aces (2) and the number of black queens (2).

$$P(E) = \frac{4}{52}$$

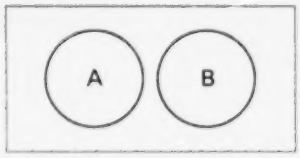
In more advanced work in probability, the concepts underlying the last two examples are summarized in the following formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



In the case that A and B are mutually exclusive events and $A \cap B = \emptyset$, then $P(A \cap B) = 0$, and the formula becomes:

$$P(A \cup B) = P(A) + P(B)$$



Problems involving variables should be included in the problems given to students.

EXAMPLE:

A fortune cookie producer has 4 different fortunes, A, B, C, and D, one of which is placed in each cookie. He has three times as many A fortunes as B and twice as many B fortunes as each of C and D. If a cookie is drawn at random from a bag of 30 cookies, find the probability that

a) it contains fortune A
b) it contains fortune B
c) it contains either fortune A or B
d) it contains either fortune C or D

Solution:

First determine that the ratio of the fortunes A:B:C:D is 6:2:1:1. If x represents the number of C fortunes, then $10x = 30$ and $x = 3$. There are 18 A fortunes, 6 B fortunes, and 3 each of C and D fortunes. Thus, the required probabilities are:

a) $\frac{18}{30}$ b) $\frac{6}{30}$ c) $\frac{24}{30}$ d) $\frac{6}{30}$

Probability of the Complement of an Event

The probability of the complement of an event is the probability that the event will not occur.

$$P(\text{not } E) = 1 - P(E)$$

EXAMPLE 1:

Find the probability that a number selected from the integers 1 to 15, inclusive, is not a prime number.

Solution:

The prime numbers in the set are 2, 3, 5, 7, 11, 13. There are 6 numbers in the set which are prime. Thus, $P(\text{prime}) = \frac{6}{15}$

Therefore, P(not prime) = 1 − P(prime)

$$P(\text{not prime}) = 1 - \frac{6}{15}$$

$$P(\text{not prime}) = \frac{9}{15}$$

EXAMPLE 2:

If the probability that it will rain on a given day is 40%, what is the probability that it will not rain?

Solution:

$$P(\text{not rain}) = 1 - \frac{40}{100} \text{ or } \frac{60}{100} \text{ or } 60\%$$

EXAMPLE 3:

If the probability that an event occurs is $\frac{x}{3}$, what is probability that the event will not occur?

Solution:

$$P(\text{not occur}) = 1 - \frac{x}{3}$$

$$= \frac{3}{3} - \frac{x}{3}$$

$$= \frac{3 - x}{3}$$

COUNTING PROCEDURES

Since we define P(A), the probability of the event A occurring, as the ratio of the number of times event A occurs to the total number of possible outcomes, success in determining P(A) is dependent upon the ability to count or otherwise determine the numbers in this ratio. Students should become familiar with a variety of techniques for counting.

Sample Space

A sample space is the set of all possible outcomes. There are several ways of specifying the sample space of an experiment, one of which is to make a list or roster.

EXAMPLE 1:

Two coins are tossed. Give the sample space.

Solution:

The sample space for this experiment is: {(H,H), (H,T), (T,H), (T,T)}.

EXAMPLE 2:

A school cafeteria offers a choice of 2 kinds of sandwiches (tuna or egg salad) and 3 beverages (white milk, chocolate milk, orange soda). What are the possible choices of a sandwich and a beverage?

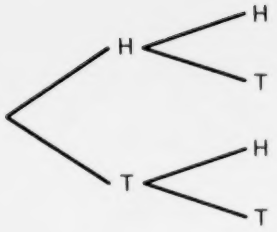
Solution:

{(tuna, white), (tuna, chocolate), (tuna, orange), (egg salad, white), (egg salad, chocolate), (egg salad, orange)}.

Rather than list the sample space, a diagram called a tree diagram can be drawn to assist in counting the total number of outcomes and the favorable outcomes.

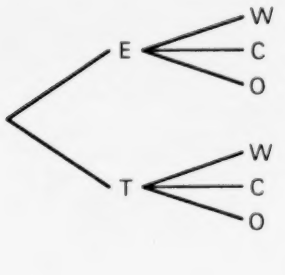
In the previous examples, the following tree diagrams could be used to organize the sample space.

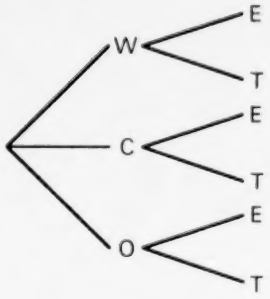
EXAMPLE 1:



EXAMPLE 2:

For this problem, either of the following trees illustrate the sample space.



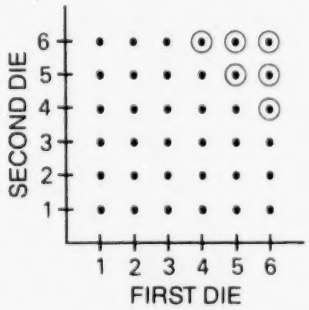


A dot diagram can sometimes be used to identify the sample space. Such a diagram can only be used to illustrate outcome sets whose elements are ordered pairs.

EXAMPLE 3:

How many possible outcomes are there if 2 fair dice are rolled? How many of these outcomes represent a sum equal to or greater than 10?

Solution:



The dot diagram shows that there are 36 possible outcomes in the sample space, and that 6 of these represent rolls in which the sum is 10 or greater.

Counting Principle

If one activity can occur in any of m ways and another in any one of n ways, then the total number of ways both activities can occur is given by m • n. This statement is called the Counting Principle.

Tree diagrams and dot diagrams are visual representations of this fundamental Counting Principle. As students become familiar with the Counting Principle, they should be able to determine the number of outcomes in an experiment without listing or picturing with a diagram, particularly in problems in which the diagram is unwieldy.

EXAMPLE 1:

Sally has 3 skirts and 4 blouses ready to wear. How many different outfits can Sally choose?

Solution:

12, since for each skirt she has a choice of 4 blouses. By the Counting Principle, $3 \cdot 4 = 12$.

EXAMPLE 2:

Suppose there are 7 roads between Troy and Utica. In how many different ways can Mr. Brown travel from Troy to Utica and return by a different route?

Solution:

He can go from Troy to Utica in any one of seven ways, but once he has chosen a route, he has a choice of only six by which to return. Hence, he may make the round trip in any of $7 \cdot 6$ or 42 ways.

Permutations

Combinatorics involves computing the number of ways r things can be chosen from a total of n things, either with regard to order, or without regard to order. If order is important, each arrangement is called a permutation. If order does not matter, each arrangement is called a combination. The major concern in Course I will be with permutations of things which are all different. The topic of combinations is not required in Course I, but some teachers may wish to examine it intuitively when teaching permutations.

EXAMPLE 1:

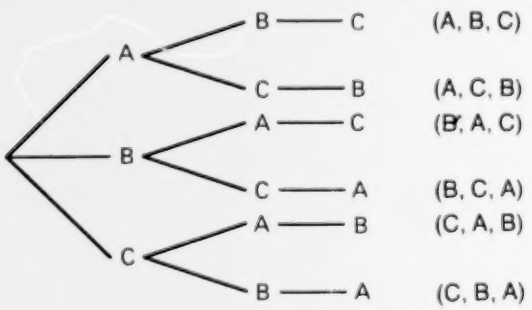
Given the letters A, B, and C, find the number of different ways to arrange them. (That is, how many permutations are there of 3 different objects?)

Solution:

The most direct approach is to list all possibilities and then count them: ABC, ACB, BAC, BCA, CBA, CAB.

A more organized approach is to note that there are 3 spaces to fill, and there are 3 alternatives for filling the first space. Once the first space is filled, there are only two remaining ones to fill the second space and then only one to fill the third space. Using the Counting Principle, we have $3 \cdot 2 \cdot 1$ or 6 ways to arrange these three letters.

A third approach is to use a tree diagram showing the resulting 6 permutations.



EXAMPLE 2:

Find the number of permutations which can be formed using the 5 letters of the word THINK.

Solution:

Using the Counting Principle, we have $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or 120 ways.

The students should see that if n different objects are to be arranged, there will be $n \cdot (n - 1) \cdot (n - 2) \ldots 2 \cdot 1$ permutations. The product $n(n - 1)(n - 2) \ldots (2)(1)$ is symbolized as n! (read, "n factorial" or "factorial n"). For example:

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24,$$
$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

The number of permutations of n different objects, using all n of them, is symbolized as ${}_nP_n = n!$

A related problem in permutations is that of forming arrangements by choosing only some of the available elements.

EXAMPLE 3:

How many arrangements can be made of 3 letters chosen from the 5 letters of the word THINK?

Solution:

In filling the first of the three spaces, we have 5 choices; for the second space, only 4 choices remain; and for the third space, only 3 choices remain. Using the Counting Principle, we have $5 \cdot 4 \cdot 3$ or 60 permutations.

When n objects are available and r ($r \leq n$) of them are going to be used in an arrangement, the number of such possible arrangements is symbolized ${}_nP_r$ (read the "permutation of n objects taken r at a time") where n denotes the number of objects and r the number selected from n. Students should see that:

$${}_nP_r = n(n - 1)(n - 2) \ldots (n - r + 1) = \frac{n!}{(n - r)!}$$

Thus, ${}_5P_3 = \dfrac{5!}{2!} = \dfrac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1}$

or $5 \cdot 4 \cdot 3 = 60$

Students should be given ample opportunities to solve problems involving permutations and the fundamental Counting Principle.

EXAMPLE 1:

In how many ways can Sally and 9 other students be seated in 10 fixed seats if Sally must sit in a certain seat because of poor vision?

Solution:

Since Sally must occupy a particular seat, there are, in effect, only 9 students to place in 9 seats. Thus, the solution is $1 \cdot {}_9P_9$.

EXAMPLE 2:

How many 4 digit numbers can be formed using the digits 1, 4, 5, 8 if repetition is not allowed?

Solution:

${}_4P_4 = 24$

EXAMPLE 3:

How many 4 digit numbers can be formed using the digits 1, 4, 5, 8 if repetition is allowed?

Solution:

Since repetition is allowed, each space may be occupied by any one of the 4 digits. Thus, the solution is $4 \cdot 4 \cdot 4 \cdot 4 = 4^4 = 256$.

EXAMPLE 4:

How many 4 digit numbers greater than 5000 can be formed using the digits 1, 4, 5, 8 if repetition is allowed?

Solution:

$2 \cdot 4^3 = 128$

PROBABILITY OF EVENTS OCCURRING JOINTLY

Extension of the Counting Principle

An extension of the Counting Principle can be applied to the probability of two or more events occurring jointly. If the probability of one event, E, is given by m ($0 \leq m \leq 1$) and the probability of another event, F, is n ($0 \leq n \leq 1$), then the probability of E and F jointly occurring is given by $m \cdot n$.

EXAMPLE 1:

A card is drawn from a standard deck, set aside, and another card is drawn. What is the probability that both cards are aces?

Solution:

P(1st card is ace) = $\frac{4}{52}$. To be successful the first card must be an ace. Assuming the success of an ace in the first drawing, there now remain 51 cards in the pack of which 3 are aces. So P(2nd card is ace) = $\frac{3}{51}$

Thus, P(both are aces) = $\frac{4}{52} \cdot \frac{3}{51}$ or $\frac{1}{221}$.

EXAMPLE 2:

Two cards are drawn from a standard deck. What is the probability they are both aces?

Solution:

Drawing two cards simultaneously is mathematically equivalent to drawing one card, not replacing it, and drawing a second card. Thus, the solution is the same as for example 1.

EXAMPLE 3:

Assuming the probability that a woman gives birth to a boy is $\frac{1}{2}$, what is the probability that in a family of 5 children, all of the children are the same gender?

Solution:

This problem requires that the children are either all boys or all girls.

P(all boys) = $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ or $\left(\frac{1}{2}\right)^5$ or $\frac{1}{32}$

Similarly, P(all girls) = $\frac{1}{32}$

P(same gender) = P(all boys) + P(all girls)

P(same gender) = $\frac{1}{32} + \frac{1}{32}$

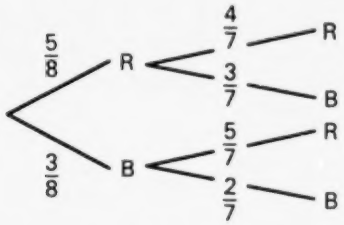
P(same gender) = $\frac{2}{32}$

EXAMPLE 4:

An urn contains five red and three black marbles. One marble is drawn, and not replaced in the urn; and then, another marble is drawn. What is the probability that both are red?

Solution:

Using the following tree diagram, the probability values on each branch, and then using our extension of the Counting Principle, we conclude that the solution is $\frac{20}{56}$.



$P(R,R,) = \frac{5}{8} \cdot \frac{4}{7} = \frac{20}{56}$

Another approach to the solution of this problem is to find the number of successes and the number of total outcomes using the Counting Principle and to form the appropriate ratio.

Number of successes $= 5 \cdot 4 = 20$.
Number of total outcomes $= 8 \cdot 7 = 56$.
P(both red) $= \frac{20}{56}$

VI.

Statistics

The statistics section in Course I is meant to reinforce and extend concepts of statistics which students have studied in K-8 mathematics. The concentration will be on methods of collecting, organizing, and interpreting data. In this course, emphasis will be on measures of central tendency; measures of dispersion will be studied in Course III.

COLLECTING AND ORGANIZING DATA

Sampling or Census

Since the essential material of statistics consists of many bits of information, or *data*, and since inferences made from the analyses of data are often of far-reaching significance, it is reasonable to investigate the adequacy of the methods by which they are collected.

In a statistical study, the choice must be made to either obtain information from every member of the group (*census*) or from a representative *sample* of the group. Frequently, it is difficult or impossible to poll the entire group, so a fair sample must be employed. Thus, some discussion of sampling techniques is in order here.

Suppose it is required to determine the number of families in Syracuse who eat Diet Woompies for breakfast. Some of the considerations which might enter into the choice of a fair, rather than a nonrepresentative, sample are:

1. Must all family members eat Diet Woompies for the family to be counted?
2. Will people be contacted by telephone? If yes, how will names and/or numbers be selected? What time of day will the calls be made? If no one is home, will the number be called again?
3. Will people be contacted by mail? If yes, how will names and/or addresses be selected?
4. Will potential respondents be interviewed? If yes, when, and where? At home? At work? On a busy street corner?
5. Will it be assumed that every respondent answers truthfully?
6. Should respondents be offered a premium for their responses?
7. What precautions will be taken to assure that the design of the questions, or the manner in asking them, will not influence the responses?

Tally, Chart, Frequency Table

When dealing with a large amount of data, it is necessary to organize it so that it can be studied efficiently. A basic technique is to prepare a tally and, from this, a frequency table. Depending on the character of the data and the purposes of the research, the tally will be made using discrete values or class intervals. The *range* (the difference between the greatest and least values) can help determine the choice of class intervals.

EXAMPLE:

At a certain college, the weights for 40 male members of the entering freshman class are as follows:

161	193	126	170	171	166	210	164
150	157	172	160	167	102	138	173
180	152	122	165	174	188	238	170
153	165	222	156	196	161	170	173
128	201	176	166	149	129	169	169

These data can be tallied and counted, and the results recorded, in a chart or table like this one:

Interval	*Tally*	*Number (frequency)*
100 – 124	//	2
125 – 149	𝍸	5
150 – 174	𝍸 𝍸 𝍸 𝍸 ///	23
175 – 199	𝍸 /	6
200 – 224	///	3
225 – 249	/	1

The choice of intervals used here is not the only one possible. The number of intervals and the interval size can reasonably be chosen in a variety of ways.

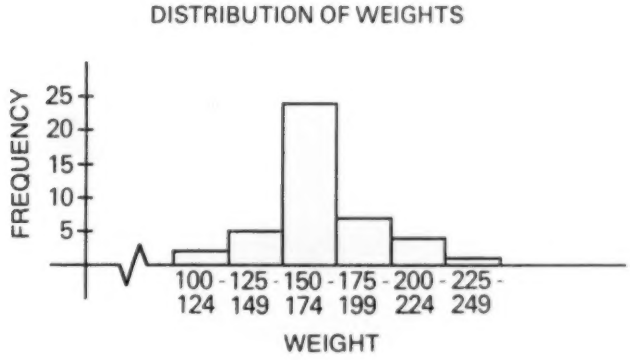
Frequency Histogram

The presentation of data, of course, is a prime factor in its usefulness. Bar graphs and circle graphs are among those with which students should already be familiar. The cooperation of social studies and business departments might be solicited at this point to provide additional examples of the variety and uses of graphs. Newspapers and magazines may also provide additional source material.

In this course, graphs known as *frequency histograms* and *cumulative frequency histograms* will be considered. A histogram is a bar graph with the bars adjacent.

EXAMPLE:

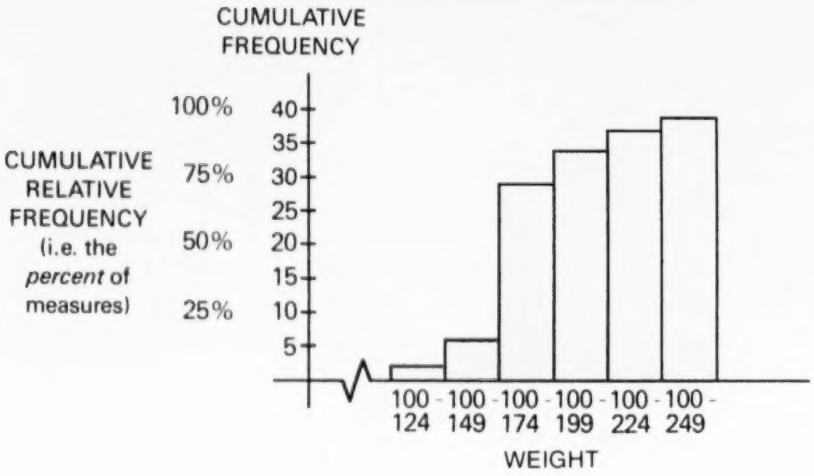
From the tally (or table) in the previous example, the following frequency histogram can be constructed.



Cumulative Frequency Histogram

At times, it is advantageous to accumulate the frequencies in a given set of data. A cumulative frequency histogram can be used to illustrate this information. Following is a cumulative frequency histogram constructed from the data in the previous example.

CUMULATIVE FREQUENCY OF WEIGHTS



MEASURES OF CENTRAL TENDENCY

A *measure of central tendency* is a single number which represents the tendency of a group of numbers to cluster, or center, around some single value. Three well-known measures of central tendency are the mean, the median, and the mode.

Mean, Median, Mode

The *mean*, or *arithmetic mean*, is the most commonly used measure of central tendency, and students will know it as the familiar "average." Its importance is well known, but there are situations where knowledge only of the mean might not be helpful in drawing useful conclusions from a set of measures. (For example, the set of measures 1, 3, 2, 4, 4, 3, 3, 500 has as its mean 65, but this number is not a useful index of central tendency for these measures.)

The *median* is the middle number when the numbers are arranged in rank order. (In the above example, the median is 3 and is a more representative value.) If there is an even number of measures, then the median is the mean of the two middle ones.

The *mode* is the measure which occurs most frequently. (The mode in the above set of numbers is 3, and in this case appears also to be a better measure of central tendency than the mean.) A set of measures may have more than one mode or no mode.

OTHER SIGNIFICANT MEASURES

Quartiles

a) The *upper quartile* of a set of measures is the measure at or below which 75% of the measures fall.

b) The *lower quartile* is the measure at or below which 25% of the measures fall.

c. The middle, or second, quartile is the median.

Percentiles

It is also possible to determine *percentile* values. The nth percentile is the measure at or below which n% of the measures fall.

It should be appreciated that, generally, the work in statistics in Course I is an elementary introduction to the subject, and that considerations appropriate to classroom treatment are not necessarily representative of the actual work of statisticians. For example, the definition of *median* used here provides for a method of precise calculation depending on whether the total number of measures is odd or even. Students should be reminded that, in practice, where there

may be hundreds or thousands of measures, such fine distinctions are of little importance. Thus, it is quite reasonable to say, for example, that the median salary of the employees of the Apex Company is *about* $21,500.

The extent to which the use and evaluation of statistics affect people in today's world is clearly very great. Statistical studies involving social and/or ethical issues may be of interest to students. It is suggested that where circumstances permit, students be allowed to discuss some of the nonmathematical issues raised in the use of statistical inference.

Appendix A

PROBLEM SOLVING

Problem Solving is an important component of both the K-6 and 7-8 mathematics programs. In elementary school, teachers are encouraged to have every lesson contain elements which help students develop problem-solving skills. The New York State syllabi for grades 7 and 8 recommend, in order to stress the importance of this topic throughout the entire curriculum, that problem-solving techniques be formally introduced at the start of each year. Below is a review of some of the problem-solving techniques that the students may have explored in their junior high school mathematics program.

A. General Techniques

Have the students review the five basic steps of problem solving:

1. **Read and re-read the problem to understand the given facts and what is to be found.**
2. **Organize the information – make lists, diagrams, etc.**
3. **Devise a plan for solving the problem.**
4. **Carry out the plan.**
5. **Check the solution.**

B. Various Problem-Solving Strategies

The problem-solving strategies presented in grades 7 and 8 include:

1. Working backwards

EXAMPLE 1:

Peter, Alice, Maria, and Leon share a bag of hard candy. Peter took 1/4 of the candy. Alice then took 1/3 of what was left. Next, Maria took 12 pieces. Finally, Leon took 4/5 of what was left, leaving 2 pieces. How many pieces of candy were in the bag in the beginning?

Start with the 2 pieces that were left and work back to the number of pieces of candy that were in the bag in the beginning.

If Leon took 4/5 of the candy, the 2 remaining pieces were 1/5. Therefore, there must have been 10 pieces before Leon took his share.

If Maria took 12 pieces, there were 22 pieces in the bag before Maria took her share.

If Alice took 1/3 of the candy, the 22 pieces were 2/3 of the candy, so there were 33 pieces before Alice took her share.

If Paul took 1/4, the 33 pieces were 3/4, so there were 44 pieces of candy in the beginning.

EXAMPLE 2:

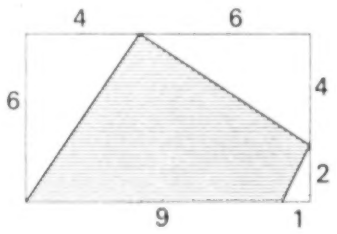
The Smiths have a teen-aged daughter who was born when Mr. Smith was in his twenties. Mr. Smith's age is now a prime number that is 2 less than a multiple of five. How old is Mr. Smith?

Mr. Smith must be between 20 + 13 and 29 + 19, that is, between 33 and 48. A number that is 2 less than a multiple of five ends in 3 or 8. Since no prime ends in 8, Mr. Smith's age must be 33 or 43. Since 33 is not a prime, Mr. Smith must be 43 years old.

2. Use of a simpler but related problem

EXAMPLE 1:

Find the area of the shaded figure enclosed in the rectangle.



Because the shaded figure is an irregular quadrilateral for which there is no familiar area formula, it is a simpler problem to find the area of that portion of the figure

that is not shaded, which consists of three right triangles. Subtract the area of the unshaded portion from the area of the rectangle.

Area of the rectangle = $6(10) = 60$

Area of the triangles = $\frac{1}{2}(6)(4) = 12$

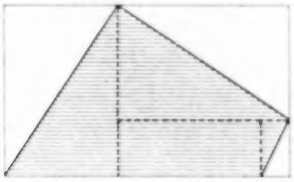
$\frac{1}{2}(6)(4) = 12$

$\frac{1}{2}(1)(2) = 1$

25

Area of the shaded portion = $60 - 25 = 35$

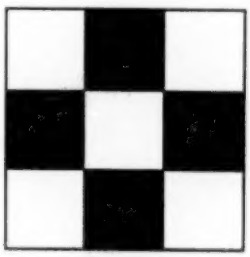
An alternate approach might be to separate the shaded portion into right triangles and a rectangle as shown below.



EXAMPLE 2:

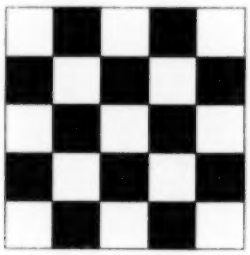
A gameboard is made up of an odd number of black and white squares with x rows of x squares, alternating black and white. Each corner is a white square. What is the difference between the number of white squares and the number of black squares?

Consider the simplest such gameboard — a 3 by 3 board.



There are 5 white and 4 black squares, so the difference is 1.

Try a 5 by 5 board.



There are 13 white and 12 black squares. Again, the difference is 1.

After a few more tries, students will decide that in all cases the difference is 1.

3. Use of patterns

EXAMPLE 1:

Find two consecutive positive integers whose squares differ by 75.

Write a table of squares and their differences. Look for a pattern.

Number	Square	Difference	Difference as a Sum
1	1		
		3	1 + 2
2	4		
		5	2 + 3
3	9		
		7	3 + 4
4	16		
		9	4 + 5
5	25		
		11	5 + 6
6	36		
		13	6 + 7
7	49		

The pattern makes it possible to solve the problem. The squares of two consecutive integers differ by the sum of the two integers.

$75 = 37 + 38$

$37^2 = 1369$

$38^2 = 1444$

$38^2 - 37^2 = 1444 - 1369 = 75$

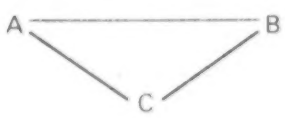
EXAMPLE 2:

Fifteen company executives gather for a business meeting. If each person shakes hands once with each other person, how many handshakes are exchanged?

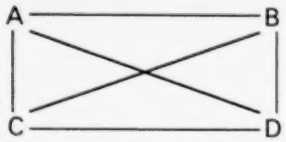
Consider two people, A and B
1 handshake is exchanged.
(A-B)

A ——— B

Consider three people, A, B, and C.
3 handshakes are exchanged.
(A-B, B-C, A-C)



Consider four people, A, B, C, and D.
6 handshakes are exchanged.
(A-B, A-C, A-D, B-C, B-D, C-D)

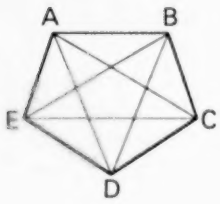


A summary so far looks like this:

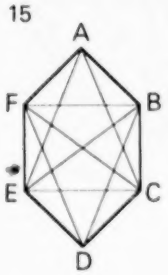
# of persons	# of handshakes
2	1
3	3
4	6

Continuing in this fashion, the following numbers will be included in the list:

5
6



10
15



Students will soon see that the increase in the number of handshakes is going up by one each time. Based on this observation, students will be able to show that 15 persons will generate 105 handshakes.

The search for a pattern can be continued until a general formula is derived showing that the number of handshakes is $\frac{n(n-1)}{2}$.

4. Use of pictures and diagrams

EXAMPLE 1:

Mr. Smith travels 50 mph for 3 hours and runs into construction which slows him down to 40 mph for the next 2 hours. Find his average rate on the trip.

A diagram is used to organize the given information.

50 mph 3 hours	40 mph 2 hours
150 miles	80 miles

The labeled diagram shows that Mr. Smith travels 230 miles in a total of 5 hours.

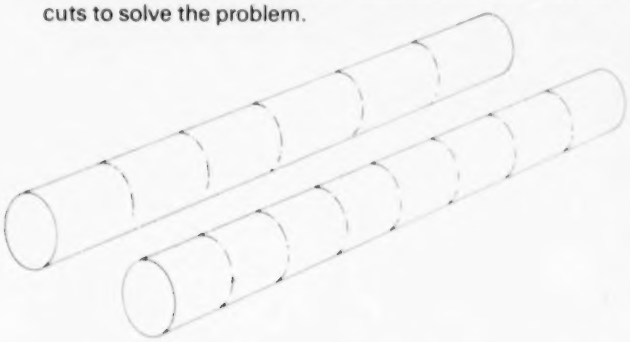
$$\text{Average rate} = \frac{\text{total distance}}{\text{total time}}$$

$$\text{Average rate} = \frac{230 \text{ mi.}}{5 \text{ hr.}} \quad 46 \text{ mi./hr.}$$

EXAMPLE 2:

A log is cut into 6 cylindrical pieces in 10 seconds. How long would it have taken to cut the log into 8 cylindrical pieces?

A diagram helps students to avoid the mistake of using the number of pieces rather than the number of cuts to solve the problem.



The diagram shows that 5 cuts are necessary to cut the log into 6 pieces. Therefore, each cut takes 2 seconds. To cut the log into 8 pieces would have required 7 cuts. Therefore, the time needed would be 14 seconds.

This problem is not correctly solved by using a proportion involving the number of pieces. However, a proportion involving the number of cuts could have been used.

5. Use of lists and charts

EXAMPLE 1:

How many two-digit primes larger than 11 and smaller than 99 are there which are still prime if the digits are reversed?

A chart showing each prime and its related number with the digits reversed will solve the problem.

Prime	Digits Reversed	Prime?
13	31	yes
17	71	yes
19	91	no
23	32	no
29	92	no
31	13	yes
etc.		

There are 8 such primes: 13, 17, 31, 37, 71, 73, 79, 97.

EXAMPLE 2:

Sam is now 24. In 6 years, he will be twice as old as his sister. How old is Sam's sister now?

By organizing this information into a chart, the answer is relatively simple to find.

	Sam	Sister
Age now	24	?
In 6 years	30	15

Since in 6 years, Sam's sister will be 15 years old, she must be 9 now.

6. Trial and error

EXAMPLE 1:

Alice has 4 more sisters than brothers. How many more sisters than brothers does her brother Tom have?

Suppose that in addition to Alice and Tom, there is 1 other boy. Then the children can be represented in the following way:

Alice	Tom	other boys	other girls
1	1	1	6

This shows that Tom has 1 brother and 7 sisters, or 6 more sisters than brothers.

Suppose there are 2 other boys. In that case, the children can be represented by:

Alice	Tom	other boys	other girls
1	1	2	7

This shows that Tom has 2 brothers and 8 sisters, or 6 more sisters than brothers.

A few more trials and students should decide that Tom has 6 more sisters than brothers in every case.

EXAMPLE 2:

Farmer Brown has horses and chickens in the barnyard. The animals have a total of 60 heads and 140 legs. How many horses and how many chickens are there in the barnyard?

Since there are 60 heads, there are 60 animals in all.

If all are chickens, there are 120 legs.
If all are horses, there are 240 legs.

If 30 are chickens and 30 are horses, there are $30 \cdot 2 + 30 \cdot 4$ or 180 legs.
Therefore, there must be more chickens.

If 40 are chickens and 20 are horses, there are $40 \cdot 2 + 20 \cdot 4$ or 160 legs.
Therefore, there must be more chickens.

If 50 are chickens and 10 are horses, there are $50 \cdot 2 + 10 \cdot 4$ or 140 legs.
Therefore, this is the required solution.

In studying the examples that are used here to illustrate problem-solving techniques, the following things should be observed.

1. The strategy used is one possible approach. Others may be possible and, perhaps, more efficient. In Example 1 above, the problem of finding two consecutive integers whose squares differ by 75 may have been solved by trial and error.
2. More than one strategy is often used in a solution. For example, diagrams are a part of the solution of both problems which use the strategy of finding a simpler but related problem.
3. The same basic principle that underlies a problem may occur in several different problem settings. For example, the problem of finding the number of handshakes exchanged among the members of a group of people is the same as the problem of finding the number of chords joining pairs of points on a circle or the problem of finding the number of games that must be played if each team in a league is to play every other team once.
4. There may be more than one basic principle that underlies the general solution of a problem. For example, the handshake problem may be considered as a problem in combinations, or it may be considered as a problem that uses the sum of consecutive integers to write the general solution since, as each new person joins the group, the number of handshakes is increased by an integer equal to the number of persons in the group before this new person was added.

Appendix B

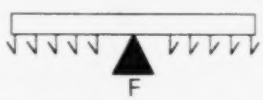
HOW DO YOU GET A GOOD RIDE ON A SEESAW?

by Bernard Fleishman

You can't sit just any place on a seesaw. If you observe people arranging themselves on a seesaw, you will notice that if they want a good balance, two people of about the same weight must sit at about the same distance from the point of balance (or *fulcrum*, as it is called). If, however, two people are of very different weights, the heavier one has to sit closer to the fulcrum. In other words the turning effect about the fulcrum depends not only on the person's weight but also on how far that person sits from the fulcrum. In general, for a given person, the further he or she sits from the fulcrum, the greater the turning effect.

A Physical Law Governing Balance

Exactly how does the turning effect depend on weight and distance? You can get an approximate idea from watching people on seesaws (and weighing them, of course — this can present practical difficulties); but an exact relationship may be found by experimenting with an apparatus (balance scale) like that shown in the figure.



A long bar or stick, balanced at its midpoint F (for fulcrum), is free to rotate about F in the plane of the paper. Suppose there are small hooks (for attaching weights) distributed equidistantly along the bar.

Now by hanging weights (one or more) on both sides of F and recording the weights and their respective locations each time a balance is achieved, you can discover the following physical law:

A weight W_1 *located to the left of F at a distance* d_1 *will be balanced by a weight* W_2 *located to the right of F at a distance* d_2 *if and only if*

$$W_1 \cdot d_1 = W_2 \cdot d_2 \qquad (1)$$

In physics, this law of balance is referred to as the "law of the lever."

Thus, the turning effect about the fulcrum F of weight W_1 located at horizontal distance d_1 from F, is measured precisely by the product $W_1 \cdot d_1$. This product is called the *torque* (pronounced "tork") or *moment* of W_1 about F. (Strictly speaking, to describe a torque completely, we must specify also its direction of rotation; for example, the weight W_1 exercises a counterclockwise torque, of magnitude $W_1 \cdot d_1$.)

From the units used for weight (say, pounds) and the units used for distance (say, feet) it follows that torques may be measured in units of pounds • feet, or "foot-pounds" (written ft.lb.), as they are usually called. For example, a half-pound weight located 8 inches to the right of F produces a torque about F in the clockwise direction, of magnitude

$$\frac{1}{2}\text{ lb} \cdot 8\text{ in} = \frac{1}{2}\text{ lb} \cdot 8\text{ in} \cdot \frac{1\text{ ft}}{12\text{ in}} = \frac{1}{3}\text{ ft.lb.}$$

Problem. A seesaw 16 feet long has its fulcrum exactly at its midpoint. Amy, a little girl weighing 50 pounds, sits at one end (say, 8 feet from the fulcrum) and asks her friend Susan, who weighs 80 pounds, to seesaw with her. For the best-balanced, smoothest ride, where should Susan sit?

SOLUTION. They will have the best balance if Susan is so located that their weights produce equal and opposite torques about the fulcrum. Therefore, if Susan sits d feet from the fulcrum (on the other side from Amy), equation (1) requires that

$$80 \cdot d = 50 \cdot 8 = 400,\ d = 5$$

Therefore, Susan should sit 5 feet from the fulcrum.

Adding Torques

From experimenting with the balance scale, you will discover another fact: Suppose two weights W_1 and W_2, are both located to the left of F, W_1 at distance d_1 and W_2 at distance d_2. Then the total counterclockwise torque produced is just the sum of the individual torques. That is, if a weight W_3 is located at distance d_3 to the right of F so that it just balances the two weights located to the left of F, then

$$(W_1 \cdot d_1) + (W_2 \cdot d_2) = W_3 \cdot d_3 \qquad (2)$$

Problem. On the scale pictured above, a 1/3-lb weight is hung 9 inches to the left of F and a 1/2-lb weight is hung 4

inches to the right of F. a) Is the bar now horizontal? If not, does it tip down on the right or the left side? b) We would like to put the scale in balance by hanging another 1/3-lb weight somewhere. Should it be hung on the left side or the right side? And how far from the fulcrum should it be hung?

SOLUTION. a) The first weight produces a c.c.w. (counter-clockwise) torque of 1/3 lb • 9/12 ft = 1/4 ft.lb.; the second produces a c.w. (clockwise) torque of 1/2 lb • 4/12 ft = 1/6 ft.lb. The scale tips down to the left because the c.c.w. torque has the greater magnitude. b) Clearly the third weight should be located to the right of F to provide additional c.w. torque. Let d feet represent the distance (to the right of F) where the third weight is hung; the c.w. torque produced by this weight has magnitude 1/3 lb x d ft. If d is such that the three weights keep the scale in balance, then

$$\frac{1}{4} = \frac{1}{6} + (\frac{1}{3} \cdot d)$$

(total c.c.w. torque must equal total c.w. torque). Then

$$\frac{d}{3} = (\frac{1}{4} - \frac{1}{6}) = \frac{1}{12}, \; d = \frac{1}{4}$$

Thus the third weight should be hung 1/4 foot or 3 inches to the right of F.

Appendix C

STUDENTS WITH HANDICAPPING CONDITIONS

The Board of Regents, through revising the Part 100 Regulations of the Commissioner and the Action Plan, has made a strong commitment to integrating the education of students with handicapping conditions into the total school program. According to Section 100.2(s) "Each student with a handicapping condition, as such term is defined in Section 200.1(ii) of this Chapter, shall have access to the full range of programs and services set forth in this Part to the extent that such programs and services are appropriate to such student's special educational needs." Districts must have policies and procedures in place to make sure that students with handicapping conditions have equal opportunities to access diploma credits, courses, and requirements.

The majority of students with disabilities have the intellectual potential to master the curricula content requirements for a high school diploma. Most students who require special education attend regular education classes in conjunction with specialized instruction and/or related services. These students must attain the same academic standards as their nonhandicapped peers in order to meet these requirements. For this reason, it is very important that at all grade levels students with handicapping conditions receive instruction in the same content areas so as to receive the same informational base that will be required for proficiency on statewide testing programs and diploma requirements.

The teacher providing instruction through this syllabus/curriculum has the opportunity to provide an educational setting which will enable the students to explore their abilities and interests. Instruction could be provided to handicapped students either by teachers certified in this subject area or by special education teachers. Teachers certified in this subject area would be providing instruction to handicapped students who are recommended by the Committee on Special Education (CSE) as being able to benefit from instruction in a regular educational setting and are appropriately placed in this setting. Special education teachers may also provide this instruction to a class of students with handicapping conditions in a special class setting.

Teachers certified in the subject area should become aware of the needs of those students with handicapping conditions participating in their classes. Instructional techniques and materials must be modified to the extent appropriate to provide students with handicapping conditions the opportunity to meet diploma requirements. Information or assistance is available through special education teachers, administrators, the Committee on Special Education or a student's Individualized Education Program (IEP).

Strategies for Modifying Instructional Techniques and Materials

1. Prior to having a guest speaker or taking field trips, it may be helpful to structure the situation. Use of a checklist or a set of questions generated by the class will help students focus on relevant information. Accessibility for students with handicapping conditions should be considered when field trips are arranged.
2. The use of computer software may be appropriate for activities that require significant amounts of writing by students.
3. Students with handicapping conditions may use alternative testing techniques. The needed testing modifications must be identified in the student's Individualized Education Program. Both special and regular education teachers need to work in close cooperation so that the testing modifications can be used consistently throughout the student's program.
4. Identify, define and pre-teach key vocabulary. Many terms in this syllabus are specific and may need continuous reinforcement for some students with handicapping conditions. it would also be helpful to provide a list of these key words to the special education teacher in order to provide additional reinforcement in the special educational setting.
5. Check periodically to determine student understanding of lectures, discussions, demonstrations, etc. and how this is related to the overall topic. Encourage students to express their understanding. It may be necessary to have small group discussions or work with partners to determine this.

6. Provide students and special education teachers with a tape of lectures that contain substantial new vocabulary content and of guest speakers for further review within their special education classes.
7. Assign a partner for the duration of a unit to a student as an additional resource to facilitate clarification of daily assignments, timelines for assignments and access to daily class notes.
8. When assigning long-term projects/reports, provide a timeline with benchmarks as indicators for completion of major project/report sections. Students who have difficulty with organizational skills and time sequence may need to see completion of sections to maintain the organization of a lengthy project/report.

Special education teachers providing this instruction must also become familiar with the goals and objectives of the curriculum. It is important that these teachers provide their students with the same or equivalent information contained in the curriculum.

Regardless of who provides the instruction, the cooperation between teachers of regular and special education programs is essential. It is important for the students as well as the total school environment.

Alternative Testing Techniques

Another consideration in assisting students with handicapping conditions to meet the requirements of regular education is the use of alternative testing techniques. Alternative testing techniques are modifications of testing procedures or formats which provide students with handicapping conditions equal opportunity to participate in testing situations. Such techniques provide the opportunity to demonstrate mastery of skills and attainment of knowledge without being limited or unfairly restricted by the existence of a handicapping condition.

The Committee on Special Education (CSE) is responsible for identifying and documenting a student's need for alternative testing techniques. This determination is made when a student is initially referred to CSE, is reviewed annually for as long as the student receives special education services, and is reviewed when the student is determined to no longer need special education services. The modifications are to be used consistently throughout the student's educational program. Principals ensure that students who have been identified by CSE as educationally handicapped are provided the alternative testing techniques which have been recommended by CSE and approved by the Board of Education.

Alternative testing techniques which have been specified on student IEPs for use by a student must be used consistently in both special and regular education settings. Regular classroom teachers should be aware of possible alternative testing techniques and should be skilled in their implementation.

The coordination and cooperation of the total school program will assist in providing the opportunity for a greater number of handicapped students to meet the requirements needed to pursue a high school diploma. The integrated provision of regular education programs, special education programs, remediation, alternative testing techniques, modified teacher techniques and materials, and access to credit through alternatives will assist in enabling such students to pursue high school diplomas to a greater degree. The teacher who provides instruction through this curriculum has a unique opportunity to assist such students in their individual goals.

Additional information on alternative testing modifications is available in the manual entitled *Alternative Techniques for Students with Handicapping Conditions*, which can be obtained from:

New York State Education Department
Office for Education of Children with
Handicapping Conditions
Room 1071 Education Building Annex
Albany, NY 12234

Infusing Awareness of Persons with Disabilities Through Curriculum

In keeping with the concept of integration, the following subgoal of the Action Plan was established:

> *In all subject areas, revisions in the syllabi will include materials and activities related to* generic subgoals such as problem solving, reasoning skills, speaking, capacity to search for information, the use of libraries and increasing student awareness of and *information about the disabled*.

The purpose of this subgoal is to ensure that appropriate activities and materials are available to increase student awareness of disabilities and issues in regard to disabilities.

This curriculum, by design, includes information, activities and materials regarding persons with handicapping conditions. Teachers are encouraged to include other examples as may be appropriate to their classrooms or the situation at hand. Teachers are also encouraged to assess the classroom environment to determine how the environment may contribute to student awareness of persons with disabilities.